A BRIEF INTRODUCTION TO DE RHAM COHOMOLOGY

Suppose that A is an open subset of $\mathbb{R}^n$ and let $\Omega^k A$ denote the $\mathbb{R}$-vector space of differential k-forms on A. (This notation is more standard than the $\Lambda^k(A)$ favored by the textbook.) Recall that the exterior derivative $d : \Omega^k A \to \Omega^{k+1} A$ is nilpotent: $d \circ d = 0$. As such, every exact form ($\omega = d\lambda$) is closed (has derivative 0).

Let $Z^k A := \ker(d : \Omega^k A \to \Omega^{k+1} A)$ be the vector space of closed forms, and let $B^k A := \mathrm{im}(d : \Omega^{k-1} A \to \Omega^k A)$ be the vector space of exact forms. We have $B^k A \subseteq Z^k A$. De Rham cohomology studies closed forms *up to* exact forms. In order to make this notion precise, we need to define quotient vector spaces.

1. Quotient vector spaces

Definition 1.1. Suppose U is a sub-vector space of V. Then the *quotient vector space of V by U* is $V/U := \{v + U \mid v \in V\}$ where $v + U = \{v + u \mid u \in U\}$.

Note that the elements of V/U are particular subsets of V called *cosets*. To ease notation, write $[v]$ for $v + U$. The vector space structure on V/U is given by $[v] + [w] = [v + w]$ and $\lambda[v] = [\lambda v]$. (Check that these operations are well-defined!) The additive identity in V/U is $[0] = U$, so elements of V/U are insensitive to addition of U. This is the sense in which V/U measures "V up to U."

Example 1.2. Let $V = \mathbb{R}^2$ and let $U = \mathbb{R}e_1$. Then V/U is naturally isomorphic to $\mathbb{R}e_2$ via the map taking $[(x, y)] \mapsto y$. (Check this!)

Example 1.3. Forget for a moment that the integers $\mathbb{Z}$ do not form a vector space. Mimicking the above construction with $V = \mathbb{Z}$ and $U = n\mathbb{Z}$ (the multiples of n), one recovers the integers modulo n, $\mathbb{Z}/n\mathbb{Z}$.

There is a natural linear map $q : V \to V/U$ given by $q(v) = [v]$, and $\ker q = U$. (If $q(v) = 0$, then $v + U = U$; since $0 \in U$, $v = v + 0 \in U$.) The quotient V/U is the "largest" vector space admitting a linear map from V which annihilates U. This is made precise in the following theorem.

Theorem 1.4. *Suppose $f : V \to W$ is a linear transformation and $U \subseteq \ker f$ (i.e., $f(U) = \{0\}$). Then there exists a unique linear transformation $\tilde{f} : V/U \to W$ such that $f = \tilde{f} \circ q$.*

Proof. Any such $\tilde{f}$ must satisfy $\tilde{f}[v] = f(v)$. We must check that the assignment $[v] \mapsto f(v)$ is well-defined and linear. If $[v] = [v']$, then $v' = v + u$ for some $u \in U$, hence $f(v') = f(v) + f(u) = f(v) + 0 = f(v)$, so the assignment is well-defined. Linearity follows from linearity of f. □

The following result is called the *first isomorphism theorem*, and provides a correspondence between quotient vector spaces and images of linear transformations.

Theorem 1.5. *Suppose $f : V \to W$ is a linear transformation. Then $V/\ker f \cong \mathrm{im}\, f$ via the assignment $[v] \mapsto f(v)$.*

Proof. The assignment is well-defined and linear by the previous theorem. Surjectivity is clear since a given $f(v) \in \mathrm{im}\, f$ is hit by $[v]$. For injectivity, note that $[v] \mapsto 0$ if and only if $f(v) = 0$, *i.e.*, if and only if $v \in \ker f$ so $[v] = \ker f = 0_{V/\ker f}$. □

Corollary 1.6. *If U is a subspace of a finite dimensional vector space V, then $\dim V/U = \dim V - \dim U$.*

Proof. This follows from the rank-nullity theorem and the previous result. □

Remark 1.7. It is possible that V and U are both infinite dimensional, but $\dim V/U < \infty$. For instance, let V be the vector spaces of sequences in $\mathbb{R}$, and let U be the sub-vector space of sequences with first term 0. Then $V/U \cong \mathbb{R}$ via the map taking $[(a_i)_{i=0}^{\infty}] \mapsto a_0$.

2. De Rham cohomology of Euclidean space

We are now prepared to make our primary definition. Recall that B^kA is the space of exact forms which sits inside Z^kA, the space of closed forms. If $k < 0$, declare that $\Omega^kA = 0$, so that $B^kA = 0$ for $k \leq 0$.

Definition 2.1. The *k-th de Rham cohomology group of A* is $H^kA := Z^kA/B^kA$.

As promised, this captures the notion of closed k-forms "up to" exact k-forms. Elements of H^kA are of the form $[\omega] = \{\omega + d\lambda \mid \lambda \in \Omega^{k-1}A\}$ where $\omega \in \Omega^kA$ satisfies $d\omega = 0$.

The 0-th de Rham cohomology group captures the number of connected components of A in the following fashion.

Proposition 2.2. *Suppose A has c connected components. Then $\dim H^0A = c$.*

Proof. Since $B^0A = 0$, we have $H^0A = Z^kA/B^kA = Z^kA/0 = Z^kA = \ker(d : \Omega^0A \to \Omega^1A)$. A 0-form $f : A \to \mathbb{R}$ has derivative 0 if and only if it is locally constant, *i.e.*, constant on each connected component. Taking the value of f on each connected component provides an isomorphism between H^0A and $\mathbb{R}^c$. □

Poincaré's theorem [*CAES* Theorem 9.11.2] provides us with our first computation of de Rham cohomology in all degrees.

Theorem 2.3. *Suppose $A \subseteq \mathbb{R}^n$ is contractible. Then*

$$H^kA \cong \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

Proof. By Poincare's theorem, if $k > 0$ then every closed k-form is exact. Thus $Z^kA = B^kA$ and $H^kA = Z^kA/B^kA = 0$. Contractible spaces have a single connected component, so $H^0A \cong \mathbb{R}$. □

3. Functoriality of de Rham cohomology

We have already seen that a C^∞ map $f : A \to B$ induces the pullback linear transformation $f^* : \Omega^kB \to \Omega^kA$. This is the map which takes $\omega = \sum f_I dy_I$ to $\sum (f_I \circ f) df_I$ where $df_{(i_1,\ldots,i_k)} = df_{i_1} \wedge \cdots \wedge df_{i_k}$ and f_i is the i-th component of f. We would like for pullback to descend to a map on cohomology groups, and this is indeed the case:

Theorem 3.1. *If $f : A \to B$ is a C^∞ map, then pullback of forms induces a linear transformation*

$$f^* : H^kB \to H^kA$$

taking $[\omega] \mapsto f^*[\omega] = [f^*\omega]$. Pullback is compatible with composition of C^∞ maps in that $(g \circ f)^* = f^* \circ g^*$ whenever $g \circ f$ is a well-defined composition of C^∞ maps between open subsets of Euclidean space. Furthermore, $\mathrm{id}^* = \mathrm{id}$.

Proof. It suffices to check that $f^*Z^kB \subseteq Z^kA$ and $f^*B^kB \subseteq B^kA$. (Take a moment to unpack this and understand why it is true; you will need the first isomorphism theorem for quotient vector spaces.) Both statements follow from compatibility of pullback with differentiation ($f^* \circ d = d \circ f^*$).

Compatibility with composition follows immediately from the analogous statement on forms, and similarly for $\mathrm{id}^* = \mathrm{id}$. □

Corollary 3.2. *If $f : A \to B$ is a diffeomorphism (meaning that f is C^∞ and f admits a C^∞ two-sided inverse), then $f^* : H^k B \to H^k A$ is an isomorphism.*

Proof. Let f^{-1} denote the C^∞ inverse of f, so that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. Then

$$\mathrm{id}_{H^k B} = \mathrm{id}_B^* = (f \circ f^{-1})^* = f^* \circ (f^{-1})^*$$

and similarly

$$\mathrm{id}_{H^k A} = (f^{-1})^* \circ f^*.$$

We conclude that $(f^{-1})^*$ is a two-sided linear inverse to f^*, and hence f^* is an isomorphism, as desired. □

Corollary 3.2 tells us that de Rham cohomology is a *diffeomorphism invariant*, *i.e.*, it does not change under diffeomorphism. This makes it a powerful tool for determining when two spaces are not diffeomorphic: if they have different cohomologies, then they are genuinely different (non-diffeomorphic) spaces.

Remark 3.3. More is true. It is actually possible to extend pullback of forms to maps which are merely continuous (rather than smooth) by approximation by smooth functions. The same functoriality properties hold, and thus de Rham cohomology is a homeomorphism invariant. In fact, it is also a homotopy invariant, the weakest notion of "sameness" in topology. Proving these facts would take us quite far afield, so we won't.

4. A CRITERION FOR NONTRIVIALITY OF DE RHAM COHOMOLOGY

In Theorem 2.3, we saw that $H^k\mathbb{R}^n = 0$ for $k > 0$. Are there examples of spaces with nonzero de Rham cohomology? We certainly hope so; otherwise, de Rham cohomology would not be a very useful invariant.

Nontriviality of de Rham cohomology indicates the existence of closed forms which are not exact, *i.e.*, forms ω satisfying $d\omega = 0$ but for which no λ makes $d\lambda = \omega$. We might call such a λ an antiderivative of ω, but it is classical to call λ a *potential* for ω. Thus $H^k A \neq 0$ precisely when there is a closed k-form on A which has no potential function.

Note that being closed is a local property: it only depends on the behavior of ω in small regions around each point. In contrast, exactness is global. The potential function must be simultaneously defined on the entire domain. Efforts to locally define λ such that $d\lambda = \omega$ will always be successful by Poincaré's theorem, but this will not guarantee that λ will be a well-defined form on the entirety of A. This failure of a local-to-global construction measures, in some sense, the topological complexity of the domain. This is what makes de Rham cohomology a meaningful invariant.

Let $\mathcal{C}$ be a k-chain on A such that $\partial\mathcal{C} = 0$. (This more or less means that the image of $\mathcal{C}$ has no $(k-1)$-dimensional boundary. For example, $\mathcal{C}$ could be a singular 2-cube parametrizing the surface of a sphere.) Then for any $(k-1)$-form λ, the fundamental theorem of integral calculus tells us that

$$\int_{\mathcal{C}} d\lambda = \int_{\partial\mathcal{C}} \lambda = \int_0 \lambda = 0.$$

Suppose now that ω is a k-form such that

$$\int_{\mathcal{C}} \omega \neq 0.$$

It immediately follows that ω is not exact, from which we deduce the following theorem.

Theorem 4.1. *Suppose ω is a closed k-form on A and there exists a k-chain $\mathcal{C}$ in A such that $\partial\mathcal{C} = 0$ and $\int_{\mathcal{C}} \omega \neq 0$. Then $H^k A \neq 0$ (and $[\omega]$ is a nonzero element of $H^k A$).*

We will use this theorem to find our first nonzero cohomology groups above degree 0.

Theorem 4.2. *Fix $n > 0$ and let*

$$\eta_0 := \sum_{i=1}^{n} (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n \in \Omega^{n-1}(\mathbb{R}^n \smallsetminus \{0\})$$

where the hatted differential is omitted and $f_i(x) = x_i/|x|$. Then η_0 is a closed form representing a nonzero class in $H^{n-1}(\mathbb{R}^n \smallsetminus \{0\})$.

Proof. The reader may check by computation that $d\eta_0 = 0$. Let Φ be an $(n-1)$-cube parametrizing the unit sphere in $\mathbb{R}^n$. The reader may also check that $\int_\Phi \eta_0 \neq 0$. □

Corollary 4.3. *For all n, m, $\mathbb{R}^n \smallsetminus \{0\}$ is not diffeomorphic (or homeomorphic, or homotopic) to $\mathbb{R}^m$.*

Remark 4.4. In fact, $H^k(\mathbb{R}^n \smallsetminus \{0\})$ is $\mathbb{R}$ in degrees 0 and $n-1$, and is 0 in all other degrees. To prove this, one needs a tiny bit of homological algebra and the Mayer-Vietoris sequence, which allows one to compute $H^k A$ in terms of $H^k U$, $H^k V$, and $H^k(U \cap V)$ when $U \cup V = A$ (loosely speaking). This leads to a bevy of further non-diffeomorphism results, such as $\mathbb{R}^n \cong \mathbb{R}^m$ if and only if $n = m$. (If they are diffeomorphic, then $\mathbb{R}^n \smallsetminus \{0\}$ and $\mathbb{R}^m \smallsetminus \{0\}$ must have the same cohomology in all degrees.)

5. The Brouwer fixed point theorem

Our fledgling computations in de Rham cohomology lead to some deep and interesting theorems. Let X be a set and let $f : X \to X$ be a function. A point $x \in X$ is a *fixed point* of f :iff $f(x) = x$.

Theorem 5.1 (Brouwer fixed point theorem)**.** *Let D^n denote the unit disc $\{x \in \mathbb{R}^n \mid |x| \leq 1\}$ in $\mathbb{R}^n$. Then any continuous map $f : D^n \to D^n$ has a fixed point.*

Interpretations of Brouwer's theorem can be quite poetic. Identify the 2-dimensional disc with the top layer of coffee in your coffee cup. If you stir the coffee in such a way that vertical layers of coffee do not mix, then at any given time, there is a point that is in the same place that it started. Alternatively, D^2 can be identified with the region represented by a map. Take the map, crumple up and distort it (without tearing it), then lay it flat somewhere within the region. Some point on the distorted map sits at the location it represents in the region.

One may even think that Brouwer's result is obvious, or at least intuitive. Regardless, its cohomological proof is quite striking.

Remark 5.2. Our proof of this theorem will assume that continuous functions (in particular, continuous null-homotopies) can be approximated by C^∞ functions, allowing us to apply the Poincaré theorem. See Chapter 7 of *CAES* for a demonstration of why this is valid.

Proof of the Brouwer fixed point theorem. Suppose that f does not have a fixed point, *i.e.*, $f(x) \neq x$ for all $x \in D^n$. Let $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ denote the boundary of D^n. Define a function $g : D^n \to S^{n-1}$ taking x to the point on S^{n-1} which the ray from $f(x)$ through x hits. Note that g restricts to the identity map on $S^{n-1} \subseteq D^n$. (Such a map has a name: g is a *retraction* of D^n onto S^{n-1}.)

We now use de Rham cohomology to show that no such g can exist. Let $A = \mathbb{R}^n \smallsetminus \{0\}$ and take $r : A \to A$ to be the map defined by $r(x) = x/|x|$. There is a (continuous — see Remark 5.2) homotopy which moves x to $r(x)$ along a straight line (say for times t from -1 to 0) and then applies $g((1-t)r(x))$ for $0 \leq t \leq 1$. At the end of this process, all of A has been moved to the single point $g(0)$, so A is contractible. This contradicts Theorems 2.3 and 4.2. □