MATH 202: VECTOR CALCULUS
WEDNESDAY WEEK 6 HANDOUT

Problem 1. Suppose the cone $z^2 = x^2 + y^2$ is sliced by the plane $z = x + y + 2$, forming a conic section C. What points on C are closest to and furthest from the origin in $\mathbb{R}^3$?

(a) Sketch a picture in $\mathbb{R}^3$ of the cone, plane, and C.
(b) Sketch a picture of the relevant level sets.
(c) Set up the Lagrange multipliers system of equations relevant to this problem. [*Tip*: Optimize the squared distance function, not the distance function.] How many equations and how many unknowns do you get?
(d) Solve the system and determine which points on C are closest to and furthest from the origin.

Problem 2 (*CAES* 5.4.11). Let p and q be positive numbers satisfying the equation $1/p + 1/q = 1$. Maximize the function of $2n$ variables $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^n x_i y_i$ subject to the contraints $\sum_{i=1}^n x_i^p = 1$ and $\sum_{i=1}^n y_i^q = 1$. Derive Hölder's inequality: For all nonnegative $a_1, \ldots, a_n$, $b_1, \ldots, b_n$,

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q}.$$

(Note that this is the Cauchy-Schwarz inequality when $p = q = 2$!)

Problem 3. Consider the generic optimization problems we discussed on the first day of class. Are they amenable to the method of Lagrange multipliers? (We asked for geometric conditions satisfied when the distance between two points on (a) two disjoint surfaces, (b) disjoint surface and curve, and (c) two disjoint curves was minimized.)