MATH 202: VECTOR CALCULUS
WEDNESDAY WEEK 1 HANDOUT

Problem 1. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Prove that for each $j \in \{1, \ldots, n\}$,

$$|x_j| \le |x| \le \sum_{i=1}^{n} |x_i|.$$

Draw and label a picture in $\mathbb{R}^2$ that expresses the geometric content of these inequalities.
[*Hint*: Note that $x_j = \langle x, e_j \rangle$ and that $x = \sum_{i=1}^{n} x_i e_i$.]

Problem 2. Use vectors to show that every angle inscribed in a semicircle is right. (A semicircle is half a circle. An angle inscribed in a semicircle is the angle made by two line segments emanating from the ends of the semicircle and meeting somewhere in the interior of the semicircle.)

Problem 3. The standard inner product on $\mathbb{R}^n$ is positive definite ($\langle x, x \rangle \ge 0$ with equality iff $x = 0$), symmetric ($\langle x, y \rangle = \langle y, x \rangle$), and bilinear (linear as a function in each coordinate). In this question, we consider forms on general vectors spaces which are symmetric and bilinear, but not necessarily positive definite. (In fact, we'll work with F-vector spaces, F an arbitrary field, so the concept of $\ge$ might not even make sense!)

(a) Let V be an F-vector space and suppose $B : V \times V \to F$ is symmetric and bilinear. Prove that when the characteristic of F is not 2, B is determined by its values on the diagonal:

$$B(v, w) = \frac{1}{2}(B(v + w, v + w) - B(v, v) - B(w, w))$$

for all $v, w \in V$.

(b) Suppose $e_1, \ldots, e_n$ is an ordered basis of V and let M be the $n \times n$ matrix with entry $B(e_i, e_j)$ in the i-th row and j-th column. Show that $B(v, w) = v^T M w$ (where w is shorthand for the column vector $\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ where $w = \sum w_i e_i$ and $v^T = (v_1, \ldots, v_n)$ where $v = \sum v_i e_i$).

(c) Which $\mathbb{R}$-matrix is associated with the inner product?

(d) What properties of an $n \times n$ matrix M guarantee that $(v, w) \mapsto v^T M w$ is a symmetric bilinear form?

(e) Call vectors $v, w \in V$ B-orthogonal if $B(v, w) = 0$. In what ways is this notion similar to our usual conception of orthogonality?

(f) The hyperbolic form on $\mathbb{R}^2$ is given by $h((x_1, y_1), (x_2, y_2)) = x_1 x_2 - y_1 y_2$. Determine the matrix for h and describe geometrically what it means for v and w to be h-orthogonal.