Mathematical Social Sciences 22 (1991) 187-227
North-Holland

The voters' paradox, spin, and the Borda count

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Communicated by F.W. Roush
Received 17 October 1989
Revised 18 July 1991

Electrical engineers employ some methods of linear algebra, derived from Homology Theory, to decompose the flow of current in a complex circuit into two components. The same decomposition can be applied to a 'circuit' containing nodes representing the candidates in a multicandidate election, connected by 'wires' carrying flows of net voter preference.

In this case, the cyclic component measures the tendency towards a voters' paradox, while the cocyclic component measures the spreads in the Borda counts. When the cocyclic component is stronger, it masks the cycles in the cyclic component, and a voters' paradox is avoided; we call this 'Borda Dominance'.

Methods based on this decomposition provide a host of necessary and sufficient conditions for various degrees of transitivity of majority preference. Sen's well-known sufficiency theorem, together with some stronger theorems, are shown to depend upon a strong 'double' form of the masking phenomenon. This mathematically natural generalization of Sen's key hypothesis is revealed to be equivalent to a new, quantitative form of transitivity.

Because the approach provides fresh insight into the underlying source of the voters' paradox, it appears to represent a promising new tool in social choice theory, with applications beyond those in the current paper.

Key words: Voters' paradox; Borda count; social choice.

1. Introduction

The linear algebra of directed graphs is beginning to look promising as an analytical tool in social choice theory. When applied to the voters' paradox, the methods yield proofs of Sen's Theorem, Sen (1966), and some closely related results. Fortunately, the theorem statements, proofs, and (to a more limited extent) intuitions can be divorced from the underlying linear algebra through the introduction of the derived concepts of spin and the Borda count.

That is the approach of this paper, the first of a projected series of three. We address the material at an intermediate level of mathematical sophistication, reducing the proofs of a number of theorems, of interest to those in the field of social choice, to elementary algebra. While this treatment widens the potential audience, there is some loss of intuitive content. A second paper, intended primarily for mathematicians, will explore the foundations of the theory, which lie in an application of the boundary map of homology theory. The final paper will draw conclusions of a more normative nature, and will presume very little mathematics.

We have made several references to numbered comments in the appendix (called 'Mathematical comments' and denoted MC 1, MC 2, etc.) which do refer to the underlying linear algebra. These may be skipped without damage to continuity or completeness.

Sen's Theorem provides conditions on a profile $\mathscr{U}$ that guarantee that there is no voters' paradox. We show that the pattern of pairwise majority preferences (that is, the numerical margin, for each pair of alternatives by which one is preferred over the other) can be naturally decomposed into two parts: a part that measures the tendency towards a cycling of majorities (voters' paradox) and that has a magnitude determined by a quantity called *spin*, and a cocyclic part that has a magnitude determined by the Borda count totals of the alternatives.

When the spin is large in relation to the Borda counts, it dominates and results in a voters' paradox, while sufficiently large Borda counts mask the spin, preventing the paradox. The flavor of the results can be found by reading the statements of Theorem 3, Corollaries 5, 6, 8, and 10, and Theorem 9, which reveal that Sen's Theorem can be interpreted as a 'Borda-masking' result.

To simplify the exposition, we begin by considering only profiles in which all preference orders are strict.

2. Notation and definitions

Suppose that X is a set (of 'alternatives'), N is a set (of 'individuals'), and that for each individual $i \in N$ we have a strict preference ordering P_i of the set X of alternatives. Then the sequence

$$\mathscr{U} = \langle P_i : i \in N \rangle.$$

is called a *profile of strict preference orders*. When $\mathscr{U}$ is such a profile and x and y are any two alternatives we will let $\text{Net}_{\mathscr{U}}(x>y)$ denote the following difference:

$$\begin{bmatrix} \text{the number of individuals} \\ i \text{ who have } xP_i y \end{bmatrix} - \begin{bmatrix} \text{the number of individuals} \\ j \text{ who have } yP_j x \end{bmatrix}.$$

Thus,

if $\text{Net}_{\mathscr{U}}(x>y)>0$, then more individuals prefer x to y than y to x;

if $\text{Net}_{\mathscr{U}}(x>y)<0$, then more individuals prefer y to x than x to y;

and

if $\text{Net}_{\mathscr{U}}(x>y)=0$, then equal numbers of individuals prefer x to y and y to x.

We will use a diagram G, called a *directed graph*, with vertices labelled by the alternatives, and edges arbitrarily assigned *orientations* (preferred directions). Then, in $G_{\mathcal{U}}$, each edge will be labelled by the appropriate net preference $\text{Net}_{\mathcal{U}}(x>y)$ 'flowing' in the indicated direction.

For example, if four voters ranked three candidates r, s, and t as follows:

4 voters have preference order $\begin{bmatrix} t \\ s \\ r \end{bmatrix}$ and the fifth has preference order $\begin{bmatrix} s \\ r \\ t \end{bmatrix}$, which we will write for now as $\mathcal{U} = 4\begin{bmatrix} t \\ s \\ r \end{bmatrix},\quad 1\begin{bmatrix} s \\ r \\ t \end{bmatrix}$,

then

$$\text{Net}_{\mathcal{U}}[s>r] = 4+1 = 5$$

$$\text{Net}_{\mathcal{U}}[t>s] = 4+(-1) = 3$$

$$\text{Net}_{\mathcal{U}}[r>t] = -4+(+1) = -3,$$

producing the $G_{\mathcal{U}}$ shown in Fig. 1.

Such triangular diagrams portray only three alternatives, so in the case that there are more than three, they can represent only a piece of the overall situation. However, for most of the paper we will only be concerned with properties that apply to alternatives three at a time, so most of our diagrams will be triangles, regardless of the total number of alternatives. Strictly speaking, then, our notation for such triangles ought to be $G(s,t,r)$ or $G_{\mathcal{U}}(s,t,r)$, but the context usually obviates the need for explicit parameters indicating which specific three alternatives are being considered.

Comment. Had the *sr* edge originally been oriented in the opposite direction, the edge would be labelled $\text{Net}_{\mathcal{U}}(r>s)$, which is $-\text{Net}_{\mathcal{U}}(s>r)$. Hence, in our example the same preference for s over r by a margin of 5 would have been indicated by an

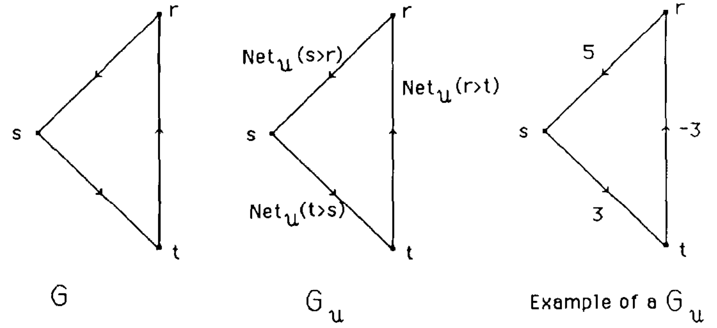


Fig. 1.

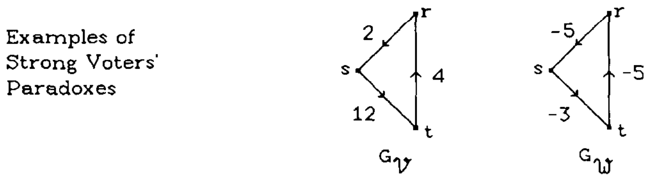


Fig. 2.

edge label of -5. Thus, while the original choice of edge orientation of G is completely arbitrary, it does serve a bookkeeping function, and consistency therefore requires that an orientation be fixed for the duration of an analysis; we will stick with the orientation of Fig. 1.

Notice that in the example in Fig. 1 most individuals prefer t to s, most prefer s to r, and most prefer t to r, so that there is no voters' paradox. A profile $\mathcal{V}$ for which the net preferences $\mathrm{Net}_{\mathcal{V}}(s>r)$, $\mathrm{Net}_{\mathcal{V}}(t>s)$ and $\mathrm{Net}_{\mathcal{V}}(r>t)$ are each strictly >0 (or a profile $\mathcal{W}$ for which each is strictly <0) will be said to have a *length three strong voters' paradox* or a *length three cycle of strict majorities*. Given our orientation of G, a strong voters' paradox occurs when either all edge labels are strictly >0, or all are strictly <0 (Fig. 2).

In these examples,

$$\mathcal{V}=7\begin{bmatrix} t \\ s \\ r \end{bmatrix},\quad 1\begin{bmatrix} t \\ r \\ s \end{bmatrix},\quad 8\begin{bmatrix} r \\ t \\ s \end{bmatrix},\quad 4\begin{bmatrix} s \\ r \\ t \end{bmatrix}$$

and

$$\mathcal{W}=5\begin{bmatrix} t \\ r \\ s \end{bmatrix},\quad 4\begin{bmatrix} r \\ s \\ t \end{bmatrix},\quad 4\begin{bmatrix} s \\ t \\ r \end{bmatrix}.$$

Similarly, a *length three weak voters' paradox* or *length three cycle of weak majorities* occurs when these same three net preferences are each ≥ 0 (or each ≤ 0), and with the same orientation of G, this occurs when either all edge labels are ≥ 0 or all are ≤ 0.

These profiles are 'paradoxical' in that no matter which of s, t, or r were to be put forward as society's first choice among these three, a majority of individuals would not only object—they could find a single alternative from among the other two which they all strictly (or weakly) prefer to society's choice.

In this paper we begin by seeking natural restrictions on a profile $\mathcal{U}$ that are sufficient and/or necessary for $\mathcal{U}$ to avoid the voters' paradoxes. There is a version of Sen's Theorem that provides such sufficient restrictions and which is free of parity considerations, such as oddness of the number of concerned voters.

In Section 9 we modify our results to account for the presence of weak preference orders, and in Section 10 we relate voters' paradox statements to their equivalent transitivity conditions. Also, a new type of transitivity condition ('transfer of preference') is introduced, and Condorcet winners are discussed. Finally, Section 11 considers parity conditions: their relationship with the original version of Sen's Theorem and with the theorems in this paper.

3. Intuition and the source of the voters' paradox

The key idea underlying this paper is that a $G_{\mathcal{U}}$ may be decomposed into two components that comprise complementary aspects of the profile $\mathcal{U}$. Figure 3 is an example of this fundamental decomposition, based on the profile

$$10\begin{bmatrix} r \\ t \\ s \end{bmatrix},\quad 2\begin{bmatrix} s \\ t \\ r \end{bmatrix},\quad 12\begin{bmatrix} t \\ s \\ r \end{bmatrix},\quad 3\begin{bmatrix} r \\ s \\ t \end{bmatrix},\quad 7\begin{bmatrix} s \\ r \\ t \end{bmatrix},\quad 8\begin{bmatrix} t \\ r \\ s \end{bmatrix}.$$

The cyclic component of any $G_{\mathcal{U}}$ is characterized (see MC 1) by having equal edge labels, while the cocyclic component is characterized by having edge labels that sum to zero. (Note that $-5\frac{1}{3}+(-7\frac{1}{3})+12\frac{2}{3}=0$.) The components are summed by adding the corresponding edge labels, so that in the $G_{\mathcal{U}}$ below (Fig. 3) $(5\frac{1}{3})+(-7\frac{1}{3})=-2$, for example.

The same type of decomposition appears in Harary (1958), for example, where it is applied to the flow of current in a circuit.

Why is this fundamental decomposition important?

(1) Every profile $\mathcal{U}$ produces a $G_{\mathcal{U}}$ that can be so decomposed.

(2) The decomposition is unique.

(3) The cocyclic component can be expressed in terms of the differences in the local Borda counts of the three alternatives s, t, and r. (The cyclic component can be similarly expressed in terms of a new quantity we call 'spin'.)

(4) The cyclic component measures the tendency towards a voters' paradox.

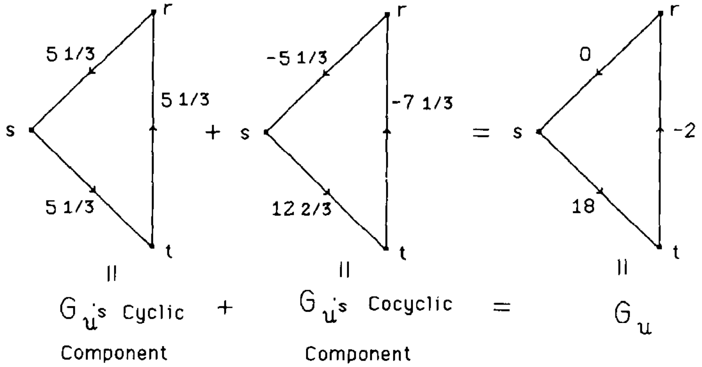


Fig. 3.

(5) Whether or not there is a voters' paradox depends on whether the cocyclic component is large enough, relative to the cyclic component, to mask the tendency towards the paradox that the cyclic component represents. (In the example at the beginning of this section, the cocyclic component is large enough to mask the cycle.)

These points are covered in the next few sections (with the exception of (2), which is established in a more general setting in Zwicker (unpublished)—see MC 2). They allow us to formulate theorems in Sections 6 and 7 that provide some necessary and sufficient conditions for omission, by a profile $\mathcal{U}$, of all length three voters' paradoxes. These conditions are all comparisons—are the Borda count 'spreads' (differences) large in comparison with the spin?

But the power of the fundamental decomposition goes beyond its role at the center of such theorems. The decomposition provides an intuitively satisfying analysis of the source of the voters' paradox.

Why does a cycle of majority preferences strike us as being paradoxical? In large part we are surprised because such cycles appear as peculiarly 'mass' effects. An individual preference order is always transitive and never has such a cycle. So how is it possible for cycles and intransitivities to arise from a group of individuals? The temptation is to believe that the cycle is a fundamentally collective phenomenon, not arising in any individual.

However, the fundamental decomposition shows that group cycles actually have their seeds in individual preference orders. It seems counter-intuitive, yet an individual preference order actually has a non-zero cyclic component or tendency towards a voters' paradox. Such a tendency is non-obvious because the cocyclic component of an individual preference order is always large enough to mask the cycle. Figure 4 shows the decomposition of the individual preference order

$$\begin{bmatrix} s \\ r \\ t \end{bmatrix}.$$

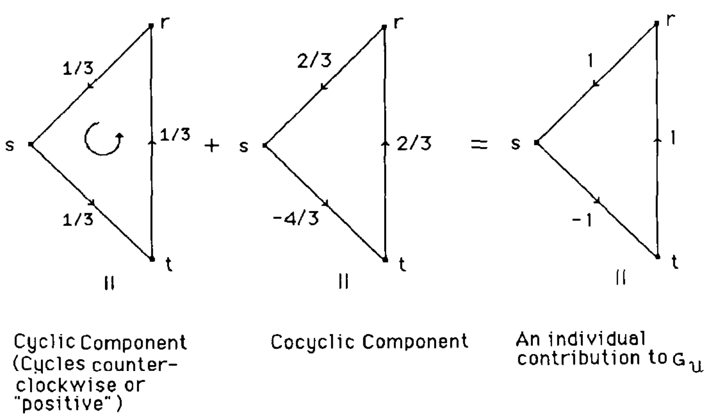


Fig. 4.

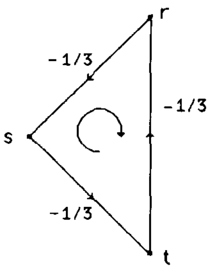


Fig. 5.

See MC 3 for a comment on the fractional values in Fig. 4. Other preference orders produce similar decompositions, but the cyclic component 'cycles' in the opposite (clockwise) direction in three out of the six possible preference orders for three alternatives (Fig. 5).

When a group of individuals produce a composite $G_{\mathcal{U}}$ by adding their contributions together, $G_{\mathcal{U}}$ can then be decomposed into a cyclic and a cocyclic component. Recall that a voters' paradox situation arises precisely when the cyclic component is predominant.

Alternatively, one can separately decompose each individual contribution first. Then the individual cyclic contributions can be added, yielding $G_{\mathcal{U}}$'s cyclic component, while the individual cocyclic contributions are summed to produce $G_{\mathcal{U}}$'s cocyclic component. *This second approach produces the same final decomposition as the first.* (See MC 2.)

As individual cyclic components are added, they reinforce each other when they are of the same sign, and cancel when of the opposite sign. The composite cyclic component is large when there is more reinforcement than cancellation. The individual cocyclic contributions similarly may reinforce or cancel (in a somewhat more complex way). Thus, a voters' paradox arises precisely when the amount of reinforcement (relative to the amount of cancellation) is greater among the cyclic components than among the cocyclic component.

In other words, each individual 'vote' consists of a cycle hidden by a mask. If the distribution of the votes falls within certain parameters, the masks tend to kill each other off while the cycles reinforce each other. The result is that the cycles (which were always present, but hidden) are revealed in the composite situation—the voters' paradox is unmasked.

4. Precise notions of spin and Borda count

In 1781, Jean-Charles de Borda introduced a social welfare function that still remains in use, and which is now referred to as the Borda count. In Borda's system a designated number of points is awarded to the alternative ranked first by an individual, a different amount is awarded to the alternative ranked second, etc. These designated amounts are called the *weights*. For each alternative, the points awarded by all the individuals are added up, these sums are then compared, and their relative sizes determine the composite preference order.

Because we are interested in properties that focus on only three alternatives at a time, we will use the local version of the Borda count that ignores an individual's preferences among the other alternatives, and only considers relative preference among the three. Thus, if the three alternatives under discussion are s, t, and r, we will say (for example) that an individual 'ranks t in second place' when we actually mean 'in second place *among* s, t, *and* r', as would be the case were s ranked third, t ranked fifth, and r ranked sixth.

For our purposes, it is convenient to use weights of 1 point for a first place finish, 0 points for a second place finish, and -1 point for each third place finish (recall that we are considering only three alternatives at this point). If x is one of the three alternatives being considered and $\mathscr{U}$ is a profile we will use $x^{\mathscr{B}}_{\mathscr{U}}$ to denote x's Borda count, or point total. Notice that

$$x^{\mathscr{B}}_{\mathscr{U}} = \begin{bmatrix} \text{the number of individuals} \\ \text{ranking } x \text{ first} \end{bmatrix} - \begin{bmatrix} \text{the number of individuals} \\ \text{ranking } x \text{ third} \end{bmatrix}.$$

When context makes it clear which profile, $\mathscr{U}$, is being considered, we will drop the subscript $\mathscr{U}$. Since $x^{\mathscr{B}}_{\mathscr{U}}$ actually depends on which three alternatives are being considered, we should actually include these alternatives as parameters in the notation ($x^{\mathscr{B}}_{\mathscr{U}}(x, y, z)$, for example), but in practice this is not needed.

Comments. (1) Any two Borda count systems in which the weights are equally spaced yield equivalent social welfare functions. The weights in the above system are equally spaced, with distances of 1 point, and the highest Borda count 'wins'. A system in which, for example, the weights for first, second, and third place are, respectively, 4, 7, and 10 points, and in which the lowest Borda count 'wins', would always determine the same composite preference order as would the system we are using.

(2) A virtue of the $-1, 0, 1$ weight distribution is that the sum of the points awarded by any individual to the three alternatives s, t, and r, under consideration, is zero. It follows easily that the Borda counts sum to zero:

$$s^{\mathscr{B}} + t^{\mathscr{B}} + r^{\mathscr{B}} = 0 \text{ (see MC 4).}$$

Given a profile $\mathscr{U}$ and three alternatives, s, t, and r, we will use $M_{\mathscr{U}}$ to denote

$\mathcal{U}$'s *spin*, and define $M_{\mathcal{U}}$ to be the sum of the 'positively circulating net preferences':

$$M_{\mathcal{U}} \equiv \text{Net}_{\mathcal{U}}(s>r) + \text{Net}_{\mathcal{U}}(r>t) + \text{Net}_{\mathcal{U}}(t>s).$$

While we might just as easily used the net preferences circulating in the reverse direction (this merely reverses $M_{\mathcal{U}}$'s sign) consistency again requires that we not switch in midstream. Notice that given our original choice of orientation of G, *the spin $M_{\mathcal{U}}$ is exactly the sum of $G_{\mathcal{U}}$'s edge labels.*

5. The six strict preference orders and their contributions

There are six possible ways to strictly order three alternatives. Table 1 assigns each of these a name and notes, for each, the amount that an individual with the preference order would contribute to $G_{\mathcal{U}}$'s edge labels, to the spin, and to each of the three Borda counts.

Explanation of Table 1. Let us trace down the entries of the first column, which is typical. If a profile were to be altered through the addition of one individual whose preference order were $\vec{u}$, what changes would result? In $\vec{u}$, r is preferred to s, so $\text{Net}_{\mathcal{U}}(s>r)$ is decreased by 1, while r is preferred to t and t to s, so both $\text{Net}_{\mathcal{U}}(r>t)$ and $\text{Net}_{\mathcal{U}}(t>s)$ are increased by 1; this explains row 3. Since s, t, and r are ranked 3rd, 2nd, and 1st, respectively, by $\vec{u}$, the point contributions to $s^{\mathcal{B}}$, $t^{\mathcal{B}}$, and $r^{\mathcal{B}}$, are -1, 0, and 1, respectively. Since $M_{\mathcal{U}}$ is just the sum of $G_{\mathcal{U}}$'s edge labels, a single vote of $\vec{u}$ contributes to the total spin $M_{\mathcal{U}}$ the sum of $\vec{u}$'s individual edge label contributions: $-1+1+1=+1$.

Table 1
Strict preference order contributions (version 1)

(1) Name of preference order	$\vec{u}$	$\vec{v}$	$\vec{w}$	$\vec{x}$	$\vec{y}$	$\vec{z}$
(2) Preference order: 1st, 2nd, 3rd	r, t, s	s, t, r	t, s, r	r, s, t	s, r, t	t, r, s
(3) Individual edge label contributions (edges s–r, r–t, t–s)	-1, 1, 1	1, -1, -1	1, -1, 1	-1, 1, -1	1, 1, -1	-1, -1, 1
(4) Individual $s^{\mathcal{B}}$ contribution	-1	$+1$	0	0	$+1$	-1
(5) Individual $t^{\mathcal{B}}$ contribution	0	0	$+1$	-1	-1	$+1$
(6) Individual $r^{\mathcal{B}}$ contribution	$+1$	-1	-1	$+1$	0	0
(7) Individual M contribution	$+1$	-1	$+1$	-1	$+1$	-1
(8) Vote count of a typical profile	a	b	c	d	e	f

The last row represents a typical profile $\mathscr{U}$ in which a denotes the number of individuals i for which P_i ranks s, t, and r as $\vec{u}$ does, b denotes the number for which P_i ranks s, t, and r as $\vec{v}$ does, etc. We will write *vote count* $(\mathscr{U})=(a,b,c,d,e,f)$, and refer to any of a,b,c,d,e, or f as *particular* vote counts.

Now Table 1 allows us to express the spin and Borda counts for $\mathscr{U}$ in terms of $\mathscr{U}$'s vote count.

Lemma 1A. *Let $\mathscr{U}$ be any profile of strict preference orders and s, t and r be three alternatives with vote count $(\mathscr{U})=(a,b,c,d,e,f)$. Then*

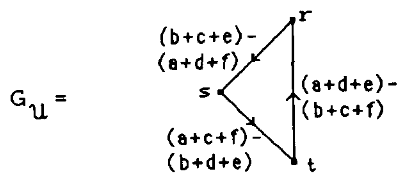
$$M_{\mathscr{U}}=(a+c+e)-(b+d+f),$$

$$s^{\mathscr{B}}_{\mathscr{U}}=(b+e)-(a+f),$$

$$t^{\mathscr{B}}_{\mathscr{U}}=(c+f)-(d+e),$$

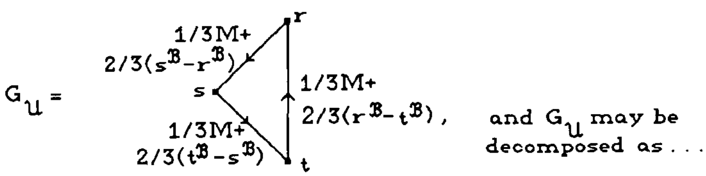
$$r^{\mathscr{B}}_{\mathscr{U}}=(a+d)-(b+c),$$

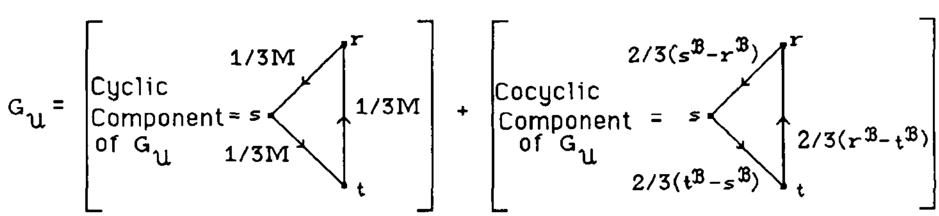
and



Proof. This is just a matter of summing individual contributions. For example, each of the a individuals with preference order $\vec{u}$ contributes $+1$ to $M_{\mathscr{U}}$ for a total contribution of $+a$, each of the b individuals with preference order $\vec{v}$ contributes -1 for a total contribution of $-b$, etc., which explains why the above formula for $M_{\mathscr{U}}$ has a '+' sign for a, a '−' sign for b, etc.

Lemma 2A (The 'fundamental decomposition'). *Let $\mathscr{U}$ be any profile of strict preference orders and s, t, and r be three alternatives. Then*





Comments and proof. First observe that when the edge labels of $G_{\mathscr{U}}$ are summed they should total M, because the three terms of $1/3M$ sum to M while the three terms built from Borda count differences sum to 0. Loosely speaking, this shows that all of $G_{\mathscr{U}}$'s spin is contained in the cyclic component of $G_{\mathscr{U}}$, while there is 0 spin in the cocyclic component of $G_{\mathscr{U}}$. The underlying linear algebra reveals that there is a *unique* way to decompose any $G_{\mathscr{U}}$ diagram into the sum of one diagram with 'pure spin' and a second diagram with zero spin—this is that decomposition.

Observe also that we are making a distinction between the spin $M_{\mathscr{U}}$, which is a number, and the cyclic component of $G_{\mathscr{U}}$, which is a labelled directed graph (strictly speaking this is the distinction between scalar and vector quantities).

The proof of Lemma 2A from Lemma 1A is straightforward, if somewhat tedious and uninformative (this is one place where the linear algebraic approach, suppressed in this paper, would yield greater insight). We have expressed M, $s^{\mathscr{B}}$, $t^{\mathscr{B}}$, and $r^{\mathscr{B}}$ in terms of $\mathscr{U}$'s vote count, (a, b, c, d, e, f). When these expressions are substituted into the quantity $1/3M + 2/3(s^{\mathscr{B}} - r^{\mathscr{B}})$, and like terms are combined, the result will be found to equal $(b+c+d)-(a+e+f)$, which checks with the $\overline{rs}$ edge label of $G_{\mathscr{U}}$; the other edges similarly check. □

We end this section with a comment on the fractions 1/3 and 2/3 that appear in Lemma 2A. These could have been eliminated had $M_{\mathscr{U}}$ been defined, in the first place, to be a third of the value we used and had the Borda weights been fixed at $+2/3$, 0, and $-2/3$ rather than $+1$, 0, and -1. Since this would merely shift the appearance of fractions within this paper, we will not bother.

However, in order to check on the presence of a voters' paradox, we need only compare the relative signs of $G_{\mathscr{U}}$'s edge labels. These relative signs are not affected when we multiply all the edge labels through by a fixed constant.

Accordingly, the conditions in the remaining sections are derived from Fig. 6.

6. Some theorems on the strong and weak voters' paradox

Figure 6 for $3G_{\mathscr{U}}$ leads immediately to

Theorem 3A(i). *Let $\mathscr{U}$ be a profile of strict preference orders. A necessary and suf-*

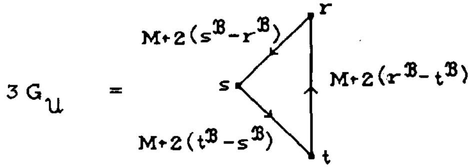


Fig. 6.

ficient condition for $\mathcal{U}$ to omit all length three strong voters' paradoxes is that for every triple s, t, r of three distinct alternatives:

$$2\min[r^{\mathcal{B}}-s^{\mathcal{B}}, t^{\mathcal{B}}-r^{\mathcal{B}}, s^{\mathcal{B}}-t^{\mathcal{B}}] \leq M \leq 2\max[r^{\mathcal{B}}-s^{\mathcal{B}}, t^{\mathcal{B}}-r^{\mathcal{B}}, s^{\mathcal{B}}-t^{\mathcal{B}}].$$

Proof. For $\mathcal{U}$ to have a strong voters' paradox either all the edge labels on $3G_{\mathcal{U}}$ must be strictly positive (>0) or all must be strictly negative (<0). Hence, for $\mathcal{U}$ to omit both types, at least one edge label must be ≤ 0, and at least one must be ≥ 0.

This can be expressed in terms of what we will call a 'Chinese menu condition': $\mathcal{U}$ omits all length three strong voters' paradoxes if and only if for each triple s, t, r of three distinct alternatives, at least one inequality is satisfied from each of the columns below:

Column 1	Column 2
$M \leq 2(r^{\mathcal{B}}-s^{\mathcal{B}})$	$M \geq 2(r^{\mathcal{B}}-s^{\mathcal{B}})$
$M \leq 2(t^{\mathcal{B}}-r^{\mathcal{B}})$	$M \geq 2(t^{\mathcal{B}}-r^{\mathcal{B}})$
$M \leq 2(s^{\mathcal{B}}-t^{\mathcal{B}})$	$M \geq 2(s^{\mathcal{B}}-t^{\mathcal{B}})$

Now M satisfies at least one of the column 1 inequalities if and only if $M \leq 2\max[r^{\mathcal{B}}-s^{\mathcal{B}}, t^{\mathcal{B}}-r^{\mathcal{B}}, s^{\mathcal{B}}-t^{\mathcal{B}}]$, and 'at least one from column 2' is equivalent to

$$2\min[r^{\mathcal{B}}-s^{\mathcal{B}}, t^{\mathcal{B}}-r^{\mathcal{B}}, -s^{\mathcal{B}}-t^{\mathcal{B}}] \leq M. \quad \square$$

Let us define a profile $\mathcal{U}$ to be *weakly Borda dominant* if for every triple s, t, r of three distinct alternatives,

$$2\min[r^{\mathcal{B}}-s^{\mathcal{B}}, t^{\mathcal{B}}-r^{\mathcal{B}}, s^{\mathcal{B}}-t^{\mathcal{B}}] \leq M \leq 2\max[r^{\mathcal{B}}-s^{\mathcal{B}}, t^{\mathcal{B}}-r^{\mathcal{B}}, s^{\mathcal{B}}-t^{\mathcal{B}}],$$

and to be *strictly Borda dominant* if for every triple s, t, r of three distinct alternatives,

$$2\min[r^{\mathcal{B}}-s^{\mathcal{B}}, t^{\mathcal{B}}-r^{\mathcal{B}}, s^{\mathcal{B}}-t^{\mathcal{B}}] < M < 2\max[r^{\mathcal{B}}-s^{\mathcal{B}}, t^{\mathcal{B}}-r^{\mathcal{B}}, s^{\mathcal{B}}-t^{\mathcal{B}}].$$

Then we may restate:

Theorem 3A(i). *A profile of strict preference orders omits all length three strong voters' paradoxes if and only if the profile is weakly Borda dominant.*

Also, we can add

Theorem 3A(ii). *A profile of strict preference orders omits all length three weak voters' paradoxes if and only if the profile is strictly Borda dominant.*

The proof of Theorem 3A(ii) is almost exactly the same as the proof of Theorem 3A(i).

Sen's sufficient condition (see Theorem 9) for omitting the strong voters' paradoxes is posed in terms of the vote counts a, b, c, d, e, and f. Before we begin translating spin and Borda count conditions into vote count conditions, it is worth asking whether there are 'improved' versions of Theorem 3. There is a well-known trade-off in which one can buy theorems with simpler and/or more natural-seeming conditions, but one has to pay by giving up the simultaneous necessity and sufficiency of the conditions, so the 'improvements' are simpler but weaker.

One of the most appealing of such improvements is posed in terms of the absolute values $|r^{\mathscr{B}}|$, $|s^{\mathscr{B}}|$, and $|t^{\mathscr{B}}|$ of the Borda counts. Given three distinct alternatives s, t, and r, let us define *$\mathscr{B}$-max* to be the greatest of these absolute values, *$\mathscr{B}$-mid* to be the one in the middle, and *$\mathscr{B}$-min* to be the least.

Corollary 4A. *Let $\mathscr{U}$ be a profile of strict preference orders. If, for every three distinct alternatives s, t, and r,*

(i) $|M| \leq 2[\mathscr{B}\text{-}max + \mathscr{B}\text{-}min]$, *then $\mathscr{U}$ omits all length three strong voters' paradoxes,*

(ii) $|M| < 2[\mathscr{B}\text{-}max + \mathscr{B}\text{-}min]$, *then $\mathscr{U}$ omits all length three weak voters' paradoxes.*

While if for some three distinct alternatives s, t, and r,

(iii) $|M| > 2[\mathscr{B}\text{-}max + \mathscr{B}\text{-}mid]$, *then $\mathscr{U}$ has a length three strong voters' paradox.*

(iv) $|M| \geq 2[\mathscr{B}\text{-}max + \mathscr{B}\text{-}mid]$, *then $\mathscr{U}$ has a length three weak voters' paradox.*

A further increase in simplicity can be achieved, at the price of greater logical 'distance' between the sufficient and necessary conditions.

Corollary 5A. *Let $\mathscr{U}$ be a profile of strict preference orders. If, for every three distinct alternatives s, t, and r,*

(i) $|M| \leq 2\max[|r^{\mathscr{B}}|, |s^{\mathscr{B}}|, |t^{\mathscr{B}}|]$, *then $\mathscr{U}$ omits all length three strong voters' paradoxes.*

(ii) $|M| < 2\max[|r^{\mathscr{B}}|, |s^{\mathscr{B}}|, |t^{\mathscr{B}}|]$, *then $\mathscr{U}$ omits all length three weak voters' paradoxes.*

While if for some three distinct alternatives s, t, and r,

(iii) $|M| > 4\max[|r^{\mathscr{B}}|, |s^{\mathscr{B}}|, |t^{\mathscr{B}}|]$, *then $\mathscr{U}$ has a length three strong voters' paradox.*

(iv) $|M| \geq 4\max[|r^{\mathscr{B}}|, |s^{\mathscr{B}}|, |t^{\mathscr{B}}|]$, *then $\mathscr{U}$ has a length three weak voters' paradox.*

Proofs. Corollary 5A follows immediately from Corollary 4A. For the proof of Corollary 4A observe that there are six possible ways in which the Borda counts $r^{\mathscr{B}}$, $s^{\mathscr{B}}$, and $t^{\mathscr{B}}$ might be ordered by $\leq$. Furthermore, since $r^{\mathscr{B}} + s^{\mathscr{B}} + t^{\mathscr{B}} = 0$ it will always be the case that one of the three Borda counts will have both maximal absolute value and lie on the opposite side of 0 from the other two. Break each of the six cases into two subcases according to whether this one Borda count is positive or negative.

By checking each of the 12 resulting cases it is possible to determine that in each case, either

$$\begin{bmatrix} \max[r^{\mathscr{B}}-s^{\mathscr{B}}, t^{\mathscr{B}}-r^{\mathscr{B}}, s^{\mathscr{B}}-t^{\mathscr{B}}] = \mathscr{B}\text{-max} + \mathscr{B}\text{-mid} \\ \text{and } |\min[r^{\mathscr{B}}-s^{\mathscr{B}}, t^{\mathscr{B}}-r^{\mathscr{B}}, s^{\mathscr{B}}-t^{\mathscr{B}}]| = \mathscr{B}\text{-max} + \mathscr{B}\text{-min} \end{bmatrix}$$

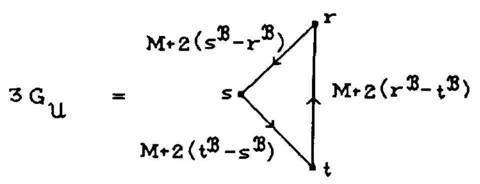
or

$$\begin{bmatrix} \max[r^{\mathscr{B}}-s^{\mathscr{B}}, t^{\mathscr{B}}-r^{\mathscr{B}}, s^{\mathscr{B}}-t^{\mathscr{B}}] = \mathscr{B}\text{-max} + \mathscr{B}\text{-min} \\ \text{and } |\min[r^{\mathscr{B}}-s^{\mathscr{B}}, t^{\mathscr{B}}-r^{\mathscr{B}}, s^{\mathscr{B}}-t^{\mathscr{B}}]| = \mathscr{B}\text{-max} + \mathscr{B}\text{-mid} \end{bmatrix}.$$

Thus, $|M| \le \mathscr{B}$-max + $\mathscr{B}$-min guarantees that M lies within the interval of Theorem 3A(i), while $|M| > \mathscr{B}$-max + $\mathscr{B}$-mid guarantees that $\mathscr{U}$ lies outside this interval, establishing parts (i) and (iii). Parts (ii) and (iv) follow similarly from Theorem 3A(ii). □

There is another corollary to Theorem 3 that is interesting in its own right, because it provides the link to Sen's Theorem and because the strengthened hypotheses turn out (see Section 10) to be necessary and sufficient conditions for some new and interesting varieties of 'quantitative' transitivity.

Recall from the proof of Theorem 3 that a necessary and sufficient condition that there be no voters' paradox among the three alternatives s, t, and r, is that $3G_{\mathscr{U}}$ have at least one negative edge label, and at least one positive one, and recall Fig. 6.



A *stronger* condition is that some *pair* of $G_{\mathscr{U}}$'s edge labels has a negative sum and some pair has a positive sum. That this is stronger follows from the fact that if a pair of numbers has a sum ≤ 0, then at least one of the numbers must be ≤ 0 (similarly for sums ≥ 0, and similarly with these weak inequalities replaced by strict ones).

Accordingly, we will define a profile $\mathscr{U}$ to satisfy the *weak edge sum condition* if for every triple of distinct alternatives s, t, and r, two of the quantities $\text{Net}_{\mathscr{U}}(s>r)$, $\text{Net}_{\mathscr{U}}(r>t)$ and $\text{Net}_{\mathscr{U}}(t>s)$ have a sum ≥ 0, and two have a sum ≤ 0; this is equivalent to saying that some pair of $G_{\mathscr{U}}$'s edge labels has a sum ≥ 0, and some pair has a sum ≤ 0, *as long as we insist* that $G_{\mathscr{U}}$'s edges be oriented cyclicly.

Similarly, $\mathscr{U}$ satisfies the *strict edge sum condition* if for every distinct s, t, and r two of $\text{Net}_{\mathscr{U}}(s>r)$, $\text{Net}_{\mathscr{U}}(r>t)$, and $\text{Net}_{\mathscr{U}}(t>s)$ have a sum <0, and two a sum >0; again this corresponds to the signs of $G_{\mathscr{U}}$'s edge label pair sums, assuming cyclical orientation.

The three possible sums obtained by adding two of the edge labels of $3G_{\mathscr{U}}$ are:

$$[M+2(t^{\mathscr{B}}-s^{\mathscr{B}})]+[M+2(r^{\mathscr{B}}-t^{\mathscr{B}})]=2M+2(r^{\mathscr{B}}-s^{\mathscr{B}}),$$

$$[M+2(s^{\mathscr{B}}-r^{\mathscr{B}})]+[M+2(t^{\mathscr{B}}-s^{\mathscr{B}})]=2M+2(t^{\mathscr{B}}-r^{\mathscr{B}}),$$

and

$$[M+2(r^{\mathscr{B}}-t^{\mathscr{B}})]+[M+2(s^{\mathscr{B}}-r^{\mathscr{B}})]=2M+2(s^{\mathscr{B}}-t^{\mathscr{B}}).$$

Thus, the weak edge sum condition is equivalent to the requirement that for every distinct triple s, t, r, at least one of the inequalities be satisfied from each of the columns below:

Column 1	Column 2
$M \le (s^{\mathscr{B}}-r^{\mathscr{B}})$	$M \ge (s^{\mathscr{B}}-r^{\mathscr{B}})$
$M \le (r^{\mathscr{B}}-t^{\mathscr{B}})$	$M \ge (r^{\mathscr{B}}-t^{\mathscr{B}})$
$M \le (t^{\mathscr{B}}-s^{\mathscr{B}})$	$M \ge (t^{\mathscr{B}}-s^{\mathscr{B}})$

As in Theorem 3, this Chinese menu condition is equivalent to boxing M inside a min–max interval. In this connection, we will define a profile $\mathscr{U}$ to be *weakly Borda double dominant* if for each triple of distinct alternatives s, t, and r we have

$$\min[s^{\mathscr{B}}-r^{\mathscr{B}}, r^{\mathscr{B}}-t^{\mathscr{B}}, t^{\mathscr{B}}-s^{\mathscr{B}}] \le M \le \max[s^{\mathscr{B}}-r^{\mathscr{B}}, r^{\mathscr{B}}-t^{\mathscr{B}}, t^{\mathscr{B}}-s^{\mathscr{B}}].$$

Also, $\mathscr{U}$ is said to be *strictly Borda double dominant* if for each triple of distinct alternatives s, t, and r we have

$$\min[s^{\mathscr{B}}-r^{\mathscr{B}}, r^{\mathscr{B}}-t^{\mathscr{B}}, t^{\mathscr{B}}-s^{\mathscr{B}}] < M < \max[s^{\mathscr{B}}-r^{\mathscr{B}}, r^{\mathscr{B}}-t^{\mathscr{B}}, t^{\mathscr{B}}-s^{\mathscr{B}}].$$

The reader should note that the interval boxing M in Borda double dominance differs from that in Borda dominance in two ways: the factor of 2 has been dropped, and the Borda count differences are now taken in reverse order. The net effect is that the interval of Borda dominance has been flipped about 0 and shrunk by a factor of 2, making the new condition roughly twice as stringent.

The above discussion constitutes the proof of the following theorem:

Theorem 6A. *Let $\mathscr{U}$ be any profile of strict preference orders. Then*

(i) *weak Borda double dominance is a necessary and sufficient condition for $\mathscr{U}$ to satisfy the weak edge sum condition, and is a sufficient condition for $\mathscr{U}$ to omit all strong voters' paradoxes, and*

(ii) *strict Borda double dominance is a necessary and sufficient condition for $\mathscr{U}$ to satisfy the strict edge sum condition, and is a sufficient condition for $\mathscr{U}$ to omit all weak voters' paradoxes.*

These edge sum conditions (equivalently, Borda double dominance) probably appear to be peripheral issues at this stage. Their connection to the transfer of preference principles is established in Section 10 (see the summary chart of parity-free properties of a profile $\mathscr{U}$), but their genesis actually lays in an analysis of Sen's Theorem that preceded any of the results in this paper. It turns out that there is an attractive proof of Sen's Theorem (not present in its direct form here) that proceeds by arguing that Sen's assumption of 'Value-Restricted Preferences', Sen (1966), referred to as 'Sen Coherence' in Taylor (unpublished), implies the weak edge sum condition, which in turn implies Sen transitivity.

7. Vote count conditions that forbid the strong and weak voters' paradoxes

Theorem 6 compares M with certain Borda count differences. Let us consider any one of these: $s^{\mathscr{B}} - r^{\mathscr{B}}$, for example. Which individual voters actually determine its size relative to M? Of the six strict preference orders, four contribute the same amount to $s^{\mathscr{B}} - r^{\mathscr{B}}$ as they do to M, so that if $\mathscr{U}$ contained only preference orders from among these four, $s^{\mathscr{B}} - r^{\mathscr{B}}$ would equal M. Of the other two preference orders, one contributes three more to M than to $s^{\mathscr{B}} - r^{\mathscr{B}}$, while the other contributes three more to $s^{\mathscr{B}} - r^{\mathscr{B}}$ than to M. Thus, given a profile $\mathscr{U}$ of strict preference orders, the question of whether M is $<$, $=$, or $> s^{\mathscr{B}} - r^{\mathscr{B}}$ is determined solely by which of these two orders achieves a higher vote count. The same is true for the other two Borda count differences in Theorem 6, as we can see from Table 2.

Explanation of Table 2. For example, a single 'vote' for $\vec{w}$ or for $\vec{x}$ or for $\vec{y}$ or for $\vec{z}$ would contribute the same amount to M as to $s^{\mathscr{B}} - r^{\mathscr{B}}$; these amounts are $+1$, -1, $+1$, and -1, respectively. However, a single vote for $\vec{u}$ would contribute 3 more to M than to $s^{\mathscr{B}} - r^{\mathscr{B}}$, since 1 is 3 more than -2, while a single vote for $\vec{v}$ would similarly contribute 3 more to $s^{\mathscr{B}} - r^{\mathscr{B}}$ than to M.

Thus, if vote count $(\mathscr{U}) = (a, b, c, d, e, f)$, then $s^{\mathscr{B}} - r^{\mathscr{B}} \leq M$ if and only if $b \leq a$, and $M \leq s^{\mathscr{B}} - r^{\mathscr{B}}$ if and only if $a \leq b$. Similar considerations lead to the following lemma:

Lemma 7A. *In the case of a profile $\mathscr{U}$ of strict preference orders, and three alter-*

Table 2
Strict preference order contributions (version 2)

Name of preference order	$\vec{u}$	$\vec{v}$	$\vec{w}$	$\vec{x}$	$\vec{y}$	$\vec{z}$
Preference order	$\begin{pmatrix} r \\ t \\ s \end{pmatrix}$	$\begin{pmatrix} s \\ t \\ r \end{pmatrix}$	$\begin{pmatrix} t \\ s \\ r \end{pmatrix}$	$\begin{pmatrix} r \\ s \\ t \end{pmatrix}$	$\begin{pmatrix} s \\ r \\ t \end{pmatrix}$	$\begin{pmatrix} t \\ r \\ s \end{pmatrix}$
Individual M contribution	$+1$	-1	$+1$	-1	$+1$	-1
Individual $s^{\mathscr{B}} - r^{\mathscr{B}}$ contribution	$\boxed{-2}$	$\boxed{+2}$	$+1$	-1	$+1$	-1
Individual $r^{\mathscr{B}} - t^{\mathscr{B}}$ contribution	$+1$	-1	$\boxed{-2}$	$\boxed{+2}$	$+1$	-1
Individual $t^{\mathscr{B}} - s^{\mathscr{B}}$ contribution	$+1$	-1	$+1$	-1	$\boxed{-2}$	$\boxed{+2}$
Vote count of a typical profile, $\mathscr{U}$	a	b	c	d	e	f

natives s, t, and r with vote count $(\mathcal{U})=(a,b,c,d,e,f)$, *consider the following two menus:*

Menu 1: Column 1	Menu 1: Column 2	Menu 2: Column 1	Menu 2: Column 2
$s^{\mathcal{B}}-r^{\mathcal{B}}\leq M$	$M\leq s^{\mathcal{B}}-r^{\mathcal{B}}$	$b\leq a$	$a\leq b$
$r^{\mathcal{B}}-t^{\mathcal{B}}\leq M$	$M\leq r^{\mathcal{B}}-t^{\mathcal{B}}$	$d\leq c$	$c\leq d$
$t^{\mathcal{B}}-s^{\mathcal{B}}\leq M$	$M\leq t^{\mathcal{B}}-s^{\mathcal{B}}$	$f\leq e$	$e\leq f$

Each inequality in Menu 1 *is equivalent to the Menu* 2 *inequality in the corresponding position. Hence the Chinese menu condition for Menu* 1 *(that at least one inequality from each column be satisfied) is equivalent to that for Menu* 2.

Now observe that the Chinese menu condition for Menu 1 is precisely the condition in Theorem 6A(i). Hence we can conclude:

Corollary 8A(i). *A necessary and sufficient condition for a profile* $\mathcal{U}$ *of strict preference orders to satisfy the weak edge sum condition (equivalently, weak Borda double dominance) and hence a sufficient condition for* $\mathcal{U}$ *to omit all length three strong voters' paradoxes, is that for every three distinct alternatives s, t, and r with associated vote count* (a,b,c,d,e,f), *at least one inequality be satisfied from each column below:*

Column 1	Column 2
$b\leq a$	$a\leq b$
$d\leq c$	$c\leq d$
$f\leq e$	$e\leq f$

A version of Lemma 7A with strict inequalities similarly yields:

Corollary 8A(ii). *A necessary and sufficient condition for a profile* $\mathcal{U}$ *of strict preference orders to satisfy the strict edge sum condition (equivalently, strict Borda double dominance) and hence a sufficient condition for* $\mathcal{U}$ *to omit all length three weak voters' paradoxes, is that for every three distinct alternatives s, t, and r with associated vote count* (a,b,c,d,e,f), *at least one inequality be satisfied from each column below:*

Column 1	Column 2
$b<a$	$a<b$
$d<c$	$c<d$
$f<e$	$e<f$

Observe that any 'null condition' such as $b=0$ (which asserts of $\mathcal{U}$ that *no* individual chooses preference order $\vec{v}$) implies one of the inequalities in Corollary

8A(i) above (but not the strict inequalities in 8A(ii)); $b=0$, for example, implies $b \leq a$ since a is necessarily non-negative. This allows us to derive the following theorem as a corollary to Corollary 8:

Theorem 9A (A version of Sen's Theorem, weaker than Corollary 8). *Let $\mathcal{U}$ be a profile of strict preference orders. A sufficient condition that $\mathcal{U}$ omit all length three strong voters' paradoxes is that for every three distinct alternatives s, t, and r with associated vote count (a,b,c,d,e,f) at least one null condition holds from each column below:*

Column 1	*Column* 2
$b=0$	$a=0$
$d=0$	$c=0$
$f=0$	$e=0$

It remains to convince the reader familiar with Sen's Theorem that the above is the parity-free, strict preference order version of this well-known result. This version states that a 'Sen coherent' (see below) profile of strict preference orders has no strong voters' paradoxes.

A profile is said to be *Sen coherent* (or *value-restricted*) if for each choice of three distinct alternatives s, t, and r, there exists at least one of the three, and one position (first, second, or third place) such that no individual ranks that alternative in that position. Sen coherence generalizes single peakedness, Black (1958), and is discussed in Brams (1985). Since there are three alternatives and three positions, $\mathcal{U}$ is Sen coherent if one of nine possible situations prevails for each choice of s, t, and r. One of these situations, for example, is that no individual rank t second. Since the only preference orders that rank t second are $\vec{u}$ and $\vec{v}$, this amounts to saying that $a=0$ and $b=0$. Observe that this is one of the nine ways that the Chinese menu condition of Theorem 9A can be satisfied. The other eight Sen coherent situations can easily be seen to correspond to the other eight ways in which we can combine one null condition from column 1 (of Theorem 9) with one from column 2. Thus, the condition of Theorem 9 is equivalent to Sen coherence.

There exists a *third* version of Theorem 6, which is worth stating because of its intuitive content. Notice that a, c, and e are the vote counts of the three preference orders ($\vec{u}$, $\vec{w}$, and $\vec{y}$, respectively) with positive spins, while b, d, and f are the vote counts of their respective *reverse* orders ($\vec{v}$, $\vec{x}$, and $\vec{z}$, respectively) each of which has negative spin.

We will define $\mathcal{U}$'s spin to be *uniformly imbalanced* for the distinct alternatives s, t, and r if there is a spin sign (either + or −) such that each vote count associated with a preference order having spin with this sign, strictly exceeds the vote count associated with its reverse order. In terms of inequalities this can be restated as

$$[a>b \text{ and } c>d \text{ and } e>f] \quad \text{or} \quad [b>a \text{ and } d>c \text{ and } f>e].$$

It is reasonable to use the term 'uniform imbalance' for this condition, since for M to be 'imbalanced' (non-zero) is equivalent to the weaker condition:

$$[(a+c+e)>(b+d+f)] \quad \text{or} \quad [(b+d+d)>(a+c+e)].$$

This allows us to state

Corollary 10 (third version of Theorem 6). *A necessary and sufficient condition for a profile $\mathscr{U}$ of strict preference orders to satisfy the weak edge sum condition, and hence a sufficient condition for $\mathscr{U}$ to omit all strong voters' paradoxes, is that for each choice of three distinct alternatives, s, t, and r, $\mathscr{U}$'s spin is* not *uniformly imbalanced.*

(Clearly we could phrase a version of Corollary 10 for the strict edge sum condition and weak voters' paradoxes, as well.)

Proof. If $\mathscr{U}$ does not have uniformly imbalanced spin, then one of the three inequalities $a>b$, $c>d$, $e>f$ must fail and one of $b>a$, $d>c$, $f>e$ must fail; this is equivalent to the Chinese menu condition holding in Corollary 8.

It should be stressed that Theorem 6, and Corollaries 8 and 10 are essentially statements of the same result, phrased differently.

Comment. In the case of precisely three alternatives, there is an interesting heuristic argument suggesting that the likelihood that any of the conditions equivalent to the weak edge sum condition hold for a 'randomly' chosen profile $\mathscr{U}$ is $>3/4$, so there is a less than 25% chance of uniformly imbalanced spin (and therefore, a less than 25% chance of $\mathscr{U}$ having a strong voters' paradox).

The argument is as follows: if a, b, c, d, e and f are chosen at random (in the sense that every vote count (a, b, c, d, e, f) summing to the number of voters is equally likely), then

$a>b$ or $b>a$ are equally likely,

$c>d$ or $d>c$ are equally likely,

and

$e>f$ or $f>e$ are equally likely.

This means that the eight possible ways of choosing a combination of one of each of two possibilities (such as $b>a$ and $d>c$ and $e>f$) are all equally likely. Since only two of these combinations yield uniformly imbalanced spin, the odds of having uniformly imbalanced spin are 2 in 8, or 25%. These calculations omit the possibility that $a=b$ or $c=d$ or $e=f$. If any of these hold, then spin is *not* uniformly imbalanced, so that the actual odds are *less* than 25%. Of course, if a, b, c, d, e, and f are chosen randomly from among a large number of possibilities (say each is chosen between 0 and 999), then such an equality is highly unlikely so that 25% becomes very close to the right odds for uniform imbalance of spin. Since the weak edge sum condition, and its equivalent, are not precisely the same as other transitivity conditions studied in the literature, this calculation will not yield the probabilities found in the literature for those other conditions.

8. Two examples

In this section we choose a profile $\mathcal{U}$ that 'barely' meets the equivalent conditions of Theorem 6 and Corollaries 8 and 10 (but not that of Sen's Theorem) and which therefore has no length three voters' paradoxes, and we work through some of the relevant quantities.

A modification of this example provides a proof that there exist profiles meeting the necessary and sufficient conditions of Theorem 3A (and which therefore have no strong voters' paradoxes) but failing to meet the condition of Theorem 6. This example serves to establish that the weak edge sum condition is *not* necessary for the omission of length three strong voters' paradoxes.

Example 1 (three alternatives in total; s, t, and r)

Preference orders:	$\vec{u}$	$\vec{v}$	$\vec{x}$	$\vec{x}$	$\vec{y}$	$\vec{z}$
Vote count ($\mathcal{U}$):	10	2	12	3	7	8
	(a)	(b)	(c)	(d)	(e)	(f)

Observe that $10>2$, $12>3$, but 7 is *not* >8, so $\mathcal{U}$'s spin is not uniformly imbalanced, hence we know that $\mathcal{U}$ omits the length three strong voters' paradoxes, by Corollary 10.

Applying Lemma 1A tells us that

$$M=(a+c+e)-(b+d+f)=29-13=16,$$

$$s^{\mathcal{B}}=(b+e)-(a+f)=9-18=-9,$$

$$t^{\mathcal{B}}=(c+f)-(d+e)=20-10=10,$$

and

$$r^{\mathcal{B}}=(a+d)-(b+c)=13-14=-1,$$

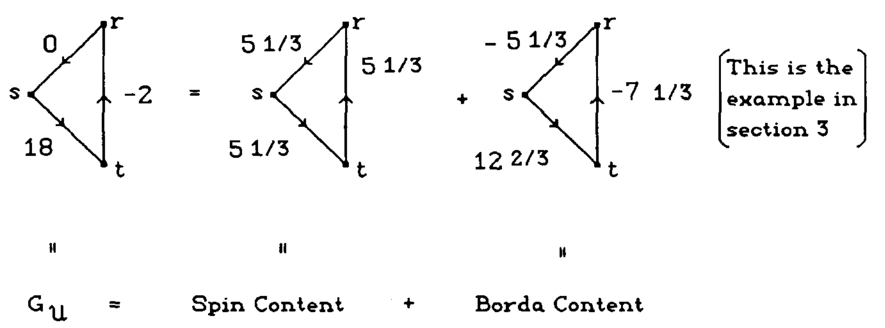


Fig. 7.

from which Lemma 2A provides the decomposition (Fig. 7) in which the Borda content may be said to 'mask' the spin content.

For Theorem 3A,

$$2(r^{\mathscr{B}} - s^{\mathscr{B}}) = 16,$$

$$2(t^{\mathscr{B}} - r^{\mathscr{B}}) = 22,$$

$$2(s^{\mathscr{B}} - t^{\mathscr{B}}) = -38.$$

The maximum of these is 22 and minimum is -38, so the Theorem 3 interval is $-38 \leq M \leq 22$.

In connection with Corollary 4 and its proof, note that in this example B-max $= 10$, B-mid $= 9$, and B-min $= 1$, so $2(B\text{-max} + B\text{-mid}) = 38$ while $2(B\text{-max} + B\text{-min}) = 22$. The interval $|M| \leq 22$ lies within the Theorem 3 interval and is the sufficient condition in Corollary 4. Also, $|M| > 38$ would guarantee that M lay outside the Theorem 3 interval and would therefore lead to a strong voters' paradox. The two conditions of Corollary 5 become, respectively, $|M| \leq 20$ and $|M| > 40$. Of course, in this example the actual value of M is 16, so M meets the sufficient conditions for the omission of the strong voters' paradox in Corollary 5 (and hence in Theorem 3 and Corollary 4, since Corollary 5 has the most stringent such condition).

We already know that in this example M will lie within the Theorem 6 interval, since lying within this interval is equivalent to *not* having uniformly imbalanced spin. The calculations are:

$$s^{\mathscr{B}} - r^{\mathscr{B}} = -8,$$

$$r^{\mathscr{B}} - t^{\mathscr{B}} = -11,$$

$$t^{\mathscr{B}} - s^{\mathscr{B}} = 19.$$

Since the minimum of these is -11 and maximum is 19 the interval becomes $-11 \leq M \leq 19$. (The doubling of this interval is $-22 \leq M \leq 38$ and has pieces that are outside of the Theorem 3 interval.) This illustrates that the factor of 2 in Theorem 3's interval bounds *must* be dropped when the Borda count differences are reversed in Corollary 6. Observe also that if this doubled Corollary 6 interval, $-22 \leq M \leq 38$, is 'reversed' to $-38 \leq M \leq 22$, the Theorem 3 interval is obtained. A quick check of the algebra reveals this to be true in general.

Comment. The 0 on the $\overline{st}$ edge of $G_{\mathscr{U}}$ has no particular significance to this example. If we jiggle $\mathscr{U}$ by changing c from 12 to 13, all the qualitative features of the example are preserved, but the number of individuals becomes odd, which rules out any 0's on $G_{\mathscr{U}}$'s edges.

Example 2. Again let s, t, and r be the only three alternatives, and consider the profile $\mathscr{V}$ with vote count $(\mathscr{V}) = (10, 2, 13, 3, 7, 6) = (a, b, c, d, e, f)$.

Observe that 10>2, 13>3, and 7>6, so that the spin is uniformly imbalanced. This means that neither Theorem 6A nor Theorem 9A (our first version of Sen's Theorem) rules out the strong voters' paradox, for this example.

However, the (doubled) Borda count differences of Theorem 3A(ii) are

$$2(r^{\mathscr{B}} - s^{\mathscr{B}}) = +10,$$

$$2(t^{\mathscr{B}} - r^{\mathscr{B}}) = 22,$$

and

$$2(s^{\mathscr{B}} - t^{\mathscr{B}}) = -32,$$

producing a min–max interval of

$$-32 < M < 22,$$

and the spin, M, is 19, which lies strictly within the interval. So Theorem 3A(ii) tells us that there are no weak voters' paradoxes.

$G_{\mathscr{V}}$ turns out to be as shown in Fig. 8.

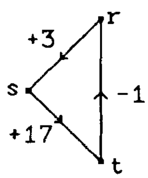


Fig. 8.

9. Weak preference orders

What happens when we allow individual preference orders in our profiles to have 'ties'?

Example. $R_j = [{}^{r}_{st}]$ (individual j is 'indifferent to s versus t').

As we will see, this change has essentially no effect on the statements of the main results of Section 6, while the Section 7 results must be changed only slightly to account for the additional preference orders; the intuitions and ideas are unchanged but the proofs of these results have to be slightly expanded.

It turns out that the only real subtlety lies in the parity-dependent results of Section 11, where the strict preference order versions are more straightforward.

In this section we discuss the changes that need to be made in the earlier material in order to account for the presence of preference orders such as R_j above.

Notation. We will refer to a profile $\mathscr{U}$ that is permitted to contain both strict preference orders and preference orders with ties, simply as a *profile*:

$$\mathscr{U} = \langle R_i : i \in N \rangle.$$

The symbol P is reserved for strict preference, while R is used for 'strictly prefers or is indifferent to', also called 'weak preference'. Thus, in our example R_j, above

$$\begin{matrix} rP_js \\ rP_jt \end{matrix} \quad \text{and} \quad \begin{matrix} sR_jt \\ tR_js \end{matrix} \qquad \text{(together, the second two assert indifference).}$$

(It would *also* be correct to state that rR_js, for example.) When we assert that a typical R_i is a *preference order* we intend that R_i denote weak preference but that the order might, or might not, actually contain any ties.

Changes in definitions. The definitions of $\text{Net}_{\mathscr{U}}(x>y)$ and of $G_{\mathscr{U}}$ do not change. There are additional possible contributions of an individual preference order to $G_{\mathscr{U}}$'s spin content and to its Borda content, however, and we have to redefine:

$$x^{\mathscr{B}}_{\mathscr{U}} = (+1)\begin{bmatrix}\text{the number of individuals}\\ \text{ranking } x \text{ alone in first}\\ \text{place}\end{bmatrix} + (+1/2)\begin{bmatrix}\text{the number ranking } x\\ \text{as tied with one other}\\ \text{in first place}\end{bmatrix}$$
$$+(-1/2)\begin{bmatrix}\text{the number ranking } x\\ \text{as tied with one other}\\ \text{in last place}\end{bmatrix} + (-1)\begin{bmatrix}\text{the number ranking } x\\ \text{alone in last place}\\ \end{bmatrix}.$$

Notice that this means that besides using the Borda count weight distribution of 1, 0, -1 for strict preference orders, we use distributions of 1/2, 1/2, -1 and of 1, $-1/2$, $-1/2$ when there is a single two-way tie for first or last place, respectively (and of 0, 0, 0 for the completely indifferent voter). Each of these weight distributions also sums to zero, so that the formula

$$s^{\mathscr{B}} + t^{\mathscr{B}} + r^{\mathscr{B}} = 0$$

remains valid. The actual weights are generated by the underlying linear algebra (see MC 4), but correspond with the common-sense proposition (sometimes followed in practice) that in a Borda count system any individual who has several alternatives in a tie awards to each the average of the weights for the tied positions.

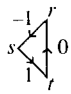
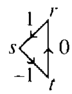
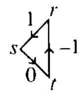
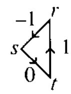
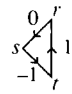
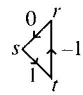
$M_{\mathscr{U}}$'s definition remains the same, as does the fact that (given our counterclockwise orientation of G), $M_{\mathscr{U}}$ is exactly $G_{\mathscr{U}}$'s edge-label sum.

For the analogues to Lemmas 1A and 2A, we need to expand our table of individual contributions (Table 3).

Since voters' paradoxes fail to arise in voting systems (such as approval voting) when there are only two intensity-of-approval levels, it is perhaps unsurprising that each of the above preference orders contribute no spin. Indeed, the fact that there is zero spin in any preference order having but two such levels may be viewed as the explanation for why approval voting yields no voters paradoxes. (Of course, there are explanations that do not refer to spin, as well.)

When we write vote count $(\mathscr{U}) = (a, b, c, d, e, f, g, h, i, j, k, l)$ we indicate that

Table 3
Weak preference order contributions

Name of preference order	$\vec{\alpha}$	$\vec{\beta}$	$\vec{\gamma}$	$\vec{\delta}$	$\vec{\varepsilon}$	$\vec{\vartheta}$
Preference order 1st	rt	s	st	r	rs	t
Preference order 2nd	s	rt	r	st	t	rs
Individual edge label contributions						
Individual $s^{\mathscr{B}}$ contribution	-1	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Individual $t^{\mathscr{B}}$ contribution	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	1
Individual $r^{\mathscr{B}}$ contribution	$\frac{1}{2}$	$-\frac{1}{2}$	-1	1	$\frac{1}{2}$	$-\frac{1}{2}$
Individual M contribution	0	0	0	0	0	0
Individual $s^{\mathscr{B}}-r^{\mathscr{B}}$ contribution	$\boxed{-1\frac{1}{2}}$	$\boxed{1\frac{1}{2}}$	$\boxed{1\frac{1}{2}}$	$\boxed{-1\frac{1}{2}}$	0	0
Individual $r^{\mathscr{B}}-t^{\mathscr{B}}$ contribution	0	0	$\boxed{-1\frac{1}{2}}$	$\boxed{1\frac{1}{2}}$	$1\frac{1}{2}$	$-1\frac{1}{2}$
Individual $t^{\mathscr{B}}-s^{\mathscr{B}}$ contribution	$\boxed{1\frac{1}{2}}$	$\boxed{-1\frac{1}{2}}$	0	0	$\boxed{-1\frac{1}{2}}$	$\boxed{1\frac{1}{2}}$
Vote count of a typical profile	g	h	i	j	k	l

$a, b, \ldots, l$ are the number of individuals who order s, t, and r according to $\vec{u}, \vec{v}, \ldots, \vec{\vartheta}$, respectively. It is now straightforward to derive:

Lemma 1B. *Let $\mathscr{U}$ be any profile of preference orders, s, t, and r be any three alternatives, and vote count $(\mathscr{U}) = (a, b, c, d, e, f, g, h, i, j, k, l)$. Then*

$$M_{\mathscr{U}} = (a+c+e)-(b+d+f),$$

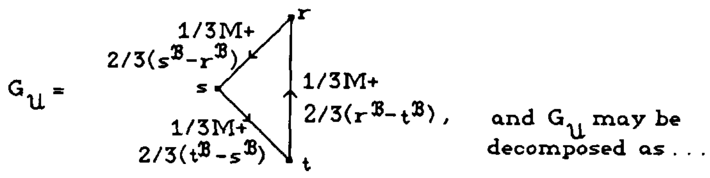
$$s^{\mathscr{B}}_{\mathscr{U}} = (b+e+h+1/2i+1/2k)-(a+f+g+1/2j+1/2l),$$

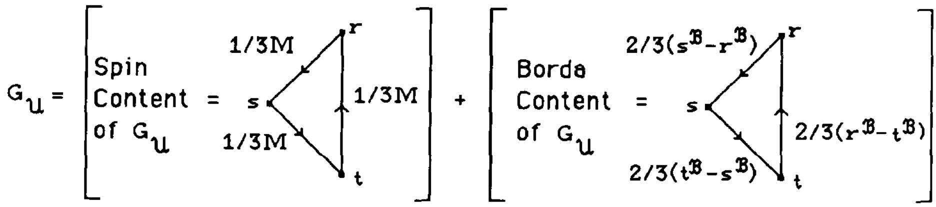
$$t^{\mathscr{B}}_{\mathscr{U}} = (c+f+l+1/2g+1/2i)-(d+e+k+1/2h+1/2j),$$

$$r^{\mathscr{B}}_{\mathscr{U}} = (a+d+j+1/2g+1/2k)-(b+c+i+1/2h+1/2l).$$

The new version of Lemma 2A for profiles (which may include weak preference orders) reads exactly as the original—perhaps this suggests that the decomposition of $G_{\mathscr{U}}$ (into a spin and a Borda part) uncovers a truth intrinsic to the situation:

Lemma 2B. *Let $\mathscr{U}$ be any profile and s, t, and r be three alternatives. Then*





The proof and comments for Lemma 2B are exactly as for Lemma 2A. Of course, the actual checking of algebra takes longer since there are additional contributions to the vote count from the new preference orders.

All the results of Section 6 go through for profiles exactly as they did for the more restrictive 'profiles of strict preference orders', because they all flow from Lemma 2B.

Theorems 3B and 6B, and Corollaries 4B and 5B. *Exactly as in the 'A' versions, with 'profile' replacing 'profile of strict preference orders'.*

The vote count conditions that forbid voters' paradoxes must be modified when we lengthen vote counts to include the new preference orders.

Recall that of the six strict preference orders for three alternatives only one, $\vec{u}$, contributes more to M than to $s^{\mathscr{B}} - r^{\mathscr{B}}$, and only $\vec{v}$ contributes more to $s^{\mathscr{B}} - r^{\mathscr{B}}$ than to M. This is why the relative sizes of a and b (the vote counts for $\vec{u}$ and $\vec{v}$) were alone enough to determine the relative sizes of $s^{\mathscr{B}} - r^{\mathscr{B}}$ and M.

Among the weak preference orders, however, $\vec{\alpha}$ and $\vec{\delta}$ contribute more to M than to $s^{\mathscr{B}} - r^{\mathscr{B}}$, while $\vec{\beta}$ and $\vec{\gamma}$ contribute more to $s^{\mathscr{B}} - r^{\mathscr{B}}$ than to M. These differences in individual contributions are half the size of the differences contributed by $\vec{u}$ and $\vec{v}$. Thus, before we had

$$s^{\mathscr{B}} - r^{\mathscr{B}} \leq M, \quad \text{if and only if } b \leq a \text{ (strict preference orders only),}$$

while now we get

$$s^{\mathscr{B}} - r^{\mathscr{B}} \leq M, \quad \text{if and only if } b + 1/2h + 1/2i \leq a + 1/2g + 1/2j$$
$$\text{(weak preference orders included).}$$

Similarly, we obtain:

Lemma 7B. *In the case of a profile and three alternatives s, t, and r with vote count* $(\mathscr{U}) = (a, b, c, d, e, f, g, h, i, j, k, l)$ *consider the following two menus:*

Menu 1

Column 1	Column 2
$s^{\mathscr{B}} - r^{\mathscr{B}} \leq M$	$M \leq s^{\mathscr{B}} - r^{\mathscr{B}}$
$r^{\mathscr{B}} - t^{\mathscr{B}} \leq M$	$M \leq r^{\mathscr{B}} - t^{\mathscr{B}}$
$t^{\mathscr{B}} - s^{\mathscr{B}} \leq M$	$M \leq t^{\mathscr{B}} - s^{\mathscr{B}}$

Menu 2

Column 1	Column 2
$b + 1/2(h+i) \leq a + 1/2(g+j)$	$a + 1/2(g+j) \leq b + 1/2(h+i)$
$d + 1/2(j+k) \leq c + 1/2(i+l)$	$c + 1/2(i+l) \leq d + 1/2(j+k)$
$f + 1/2(g+l) \leq e + 1/2(h+k)$	$e + 1/2(h+k) \leq f + 1/2(g+l)$

Each inequality in Menu 1 *is equivalent to the Menu* 2 *inequality in the corresponding position. Hence the Chinese menu condition for Menu* 1 *(that at least one inequality from each column be satisfied) is equivalent to that for Menu* 2.

As earlier, the Chinese menu condition for Menu 1 is exactly the condition of Corollary 6B(i), so we can conclude:

Corollary 8B(i) (Vote count version of Theorem 6B(i)). *A necessary and sufficient condition for a profile* $\mathscr{U}$ *to satisfy the weak edge sum condition (equivalently, weak Borda double dominance), and hence a sufficient condition for* $\mathscr{U}$ *to omit all length three strong voters' paradoxes, is that for every three distinct alternatives s, t, and r with associated vote count* $(a, b, c, d, e, f, g, h, i, j, k, l)$ *at least one inequality be satisfied from each column below:*

Column 1	Column 2
$b + 1/2(h+i) \leq a + 1/2(g+j)$	$a + 1/2(g+j) \leq b + 1/2(h+i)$
$d + 1/2(j+k) \leq c + 1/2(i+l)$	$c + 1/2(i+l) \leq d + 1/2(j+k)$
$f + 1/2(g+l) \leq e + 1/2(h+k)$	$e + 1/2(h+k) \leq f + 1/2(g+l)$

A version of Lemma 7B with strict inequalities similarly yields:

Corollary 8B(ii) (Vote count version of Theorem 6B(ii). *A necessary and sufficient condition for a profile* $\mathscr{U}$ *to satisfy the weak edge sum condition (equivalently, strict Borda double dominance), and hence a sufficient condition for* $\mathscr{U}$ *to omit all length three strong voters' paradoxes, is that for every three distinct alternatives s, t, and r with associated vote count* $(a, b, c, d, e, f, g, h, i, j, k, l)$ *at least one inequality be satisfied from each column below:*

Column 1	*Column 2*
$b+1/2(h+i)<a+1/2(g+j)$	$a+1/2(g+j)<b+1/2(h+i)$
$d+1/2(j+k)<c+1/2(i+l)$	$c+1/2(i+l)<d+1/2(j+k)$
$f+1/2(g+l)<e+1/2(h+k)$	$e+1/2(h+k)<f+1/2(g+l)$

Again, certain null conditions (such as $b=h=i=0$) imply that certain inequalities are satisfied, allowing us to derive the following theorem as a corollary to Corollary 8B(i):

Theorem 9B (A version of Sen's Theorem, weaker than Corollary 8). *Let $\mathscr{U}$ be any profile. A sufficient condition that $\mathscr{U}$ omit all length three strong voters' paradoxes is that for every three distinct alternatives s, t, and r with associated vote count $(a, b, c, d, e, f, g, h, i, j, k, l)$ at least one null condition holds from each column below*:

Column 1	*Column 2*
$b=h=i=0$	$a=g=j=0$
$d=j=k=0$	$c=i=l=0$
$f=g=l=0$	$e=h=k=0$

Discussion. Again, the nine possible combinations (of one null condition from column 1 and one from column 2) correspond to the nine situations that result in Sen coherence. For example, if no one ranks t 'among the middle' (meaning tied for first place, alone in the middle, or tied for last place), then no individual specifies preference orders $\vec{u}$, $\vec{v}$, $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$, or $\vec{\delta}$, so $a=b=g=h=i=j=0$, and the first null condition in each column is met. In checking out the other eight situations, recall that besides 'among the middle', there is 'among the best' (meaning alone in first place or tied for first) and 'among the worst' (meaning alone in last place or tied for last).

Comment on the missing Corollary 10B. It certainly would be possible to define a version of 'uniformly imbalanced spin' in the context of weak preference orders, but until such time as a satisfactory interpretation of the quantity of (for example) $b+1/2(h+i)$ can be developed, we shall not do so. In passing we observe that the preference orders $\vec{v}$, $\vec{\beta}$, and $\vec{\gamma}$ counted by b, h, and i, respectively, are:

$$\overset{\vec{v}}{\begin{bmatrix} s \\ t \\ r \end{bmatrix}}, \quad \overset{\vec{\beta}}{\begin{bmatrix} s \\ tr \end{bmatrix}}, \quad \overset{\vec{\gamma}}{\begin{bmatrix} st \\ r \end{bmatrix}}.$$

It is difficult to see how they measure any 'spin' in common, since $\vec{\beta}$ and $\vec{\gamma}$ have 0 spin. What they have in common is that they represent the three ways an individual preference order can arise from the weak order $s \geq t \geq r$ by making at least one of the two preferences strict. Also, a, g, and j (the quantities on the other side of the inequality on line 1 of the Corollary 8B menu) are counts of the three ways an individual preference order can similarly arise from the reverse order $r \geq t \geq s$. At this point these observations seem to be more labored than they are revealing, however.

10. Transitivity, voters' paradoxes, and Condorcet conditions

Up to this point we have phrased our conclusions in terms of forbidden configurations—'...then $\mathscr{U}$ has no length three strong (or weak) voters' paradoxes', meaning that for *each* choice of three distinct alternatives s, t, and r (from among all the alternatives) it is not the case that a strong (or weak) voters' paradox exists among the three. Equivalently, we may phrase this as 'no cycle of strict (or weak) majorities exists among any three of the alternatives'.

It is more traditional to phrase conclusions in terms of so-called *transitivity* conditions. For $\mathscr{U}$ any profile and x and y any alternatives let us define

$$xR_{\mathscr{U}}\,y \text{ to mean } \mathrm{Net}_{\mathscr{U}}(x>y)\geq 0,$$

and

$$xP_{\mathscr{U}}\,y \text{ to mean } \mathrm{Net}_{\mathscr{U}}(x>y)>0.$$

Thus, $xR_{\mathscr{U}}\,y$ says that among those *concerned* with x and y (i.e. those *not* indifferent to x and y) a weak majority (half or more) prefer x to y, while $xP_{\mathscr{U}}\,y$ says that a strong majority (more than half) of those concerned prefer x to y.

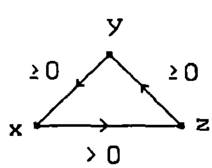
An example of a transitivity condition is

$$xR_{\mathscr{U}}\,yR_{\mathscr{U}}z \rightarrow xR_{\mathscr{U}}z \qquad (*)$$

and this is taken to mean that *for every three distinct alternatives x, y, and z*, if $xR_{\mathscr{U}}\,y$ and $yR_{\mathscr{U}}z$, then $xR_{\mathscr{U}}z$. Any such transitivity condition can be translated into a statement that certain situations are forbidden; in this case it is forbidden that $xR_{\mathscr{U}}\,y$, $yR_{\mathscr{U}}z$, and $zP_{\mathscr{U}}x$. If the edges between x, y, and z are oriented, as in Fig. 9, then this says that the combination of non-negative edge labels for the xy and yz edges with a strictly positive edge label for xz is a forbidden one.

Because each of the transitivity conditions is a statement about *every* choice of alternatives x, y, and z, the configuration shown in Fig. 10 is also ruled out.

It is easy to see that a profile $\mathscr{U}$ satisfies the transitivity condition (*) above if and only if *for each triple x, y, z of distinct alternatives and each orientation of the*



(**)

Fig. 9.

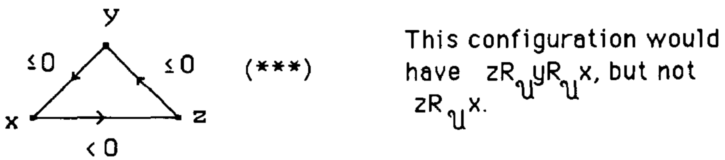


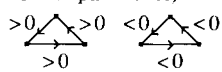
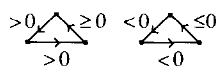
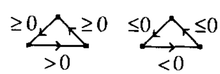
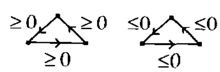
Fig. 10.

corresponding triangle G in such a way that the edges flow in a cycle, neither of the configurations (**) *or* (***) *for* $G_{\mathcal{U}}$ *occurs.*

In what follows, when we characterize various transitivity conditions in terms of forbidden configurations, the italicized words above should be understood to apply, and we shall speak simply in terms of a given transitivity condition as equivalent to certain configurations being forbidden.

As we see in Table 4, there are four qualitative transitivity conditions that seem to arise naturally and that are equivalent to certain forbidden configurations. The four are listed in order of increasing strength—that they get stronger should be clear

Table 4
Levels of qualitative transitivity and voters' paradoxes

Transitivity condition	Equivalent edge-label restrictions (assuming G is oriented in a cycle, either direction)	Equivalent forbidden configurations
(1) *Weak transitivity* $xP_{\mathcal{U}}yP_{\mathcal{U}}z \to xR_{\mathcal{U}}z$	If any two edges are >0, the third must be ≤ 0 (and if any two are <0, the third must be ≥ 0)	(no length three strong voters' paradoxes) 
(2) *Sen transitivity* $xP_{\mathcal{U}}yR_{\mathcal{U}}z \to xR_{\mathcal{U}}z$ and $xR_{\mathcal{U}}yP_{\mathcal{U}}z \to xR_{\mathcal{U}}z$[a]	If any edge is >0 and any other is ≥ 0, the third must be <0 (and if any is ≤ 0 and any other ≤ 0, the third must be ≥ 0)	
(3) *Traditional transitivity* $xR_{\mathcal{U}}yR_{\mathcal{U}}z \to xR_{\mathcal{U}}z$[b]	If any two edges are ≥ 0, the third must be ≤ 0 (and if two are ≤ 0, the third must be ≥ 0)	
(4) *Strong transitivity* $xR_{\mathcal{U}}yR_{\mathcal{U}}z \to xP_{\mathcal{U}}z$	If any two edges are ≥ 0, the third must be <0 (and if any two are ≤ 0, the third must be >0)	 (no length three weak voters' paradoxes)

[a] Equivalently, $xP_{\mathcal{U}}yP_{\mathcal{U}}z \to xP_{\mathcal{U}}z$.
[b] Equivalently, $xR_{\mathcal{U}}yP_{\mathcal{U}}z \to xP_{\mathcal{U}}z$ and $xP_{\mathcal{U}}yR_{\mathcal{U}}z \to xP_{\mathcal{U}}z$.

from the forbidden configurations, since the stronger ones forbid more configurations.

As the table indicates, 'no length three strong voters' paradoxes' is equivalent to the weakest of these conditions, while 'no length three weak voters' paradoxes' is equivalent to the strongest. Each of the two intermediate forms has a second, equivalent transitivity expression; one way to see that the equivalence holds is to observe that the 'forbidden configurations' version is equivalent to each of the expressions.

For the record we note:

Observation 11. *Let $\mathscr{U}$ be any profile (for any number of alternatives). Then the four qualitative transitivity conditions are graded in strength as*

$$\textit{Strong} \Rightarrow \textit{Traditional} \Rightarrow \textit{Sen} \Rightarrow \textit{Weak}.$$

There are other transitivity properties that are quantitative in nature in that they refer to the *amounts* by which a majority prefers one alternative to another.

One such property is of special interest because it is equivalent to the weak edge sum condition of Section 6. Because this condition is itself equivalent to weak Borda double dominance, this form of Borda dominance is revealed to be necessary and sufficient for a type of transitivity.

For context, recall that a profile $\mathscr{U}$ satisfies the weak edge sum condition if for every three distinct alternatives x, y, and z and every cycling orientation of the corresponding triangle G, $G_{\mathscr{U}}$ has two edge labels with a sum ≥ 0, and two with a sum ≤ 0.

Also recall that $\mathscr{U}$ is weakly Borda double dominant if for every three distinct alternatives s, t, and r,

$$\min[s^{\mathscr{B}} - r^{\mathscr{B}}, r^{\mathscr{B}} - t^{\mathscr{B}}, t^{\mathscr{B}} - s^{\mathscr{B}}] \leq M \leq \max[s^{\mathscr{B}} - r^{\mathscr{B}}, r^{\mathscr{B}} - t^{\mathscr{B}}, t^{\mathscr{B}} - s^{\mathscr{B}}].$$

Now we define: *$\mathscr{U}$ transfers strict preference* if for every three distinct alternatives x, y, and z and every strictly positive integer $k>0$, if $\text{Net}_{\mathscr{U}}(x>y) \geq k$ and $\text{Net}_{\mathscr{U}}(y>z) \geq k$, then $\text{Net}_{\mathscr{U}}(x>z) \geq k$.

Theorem 12. *For any profile $\mathscr{U}$, $\mathscr{U}$ satisfies the weak edge sum condition if and only if $\mathscr{U}$ transfers strict preference.*

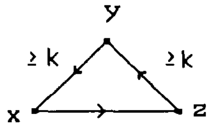


Fig. 11.

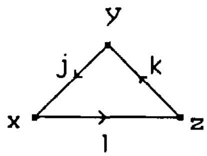


Fig. 12.

Proof. ($\Rightarrow$) Assume the weak edge sum condition. Let $k>0$ with $\text{Net}_{\mathscr{U}}(x>y)\geq k$ and $\text{Net}_{\mathscr{U}}(y>z)\geq k$. Orient G as in Fig. 11 and observe that two edges must receive edge labels $\geq k$. The sum of these two edges labels is $\geq 2k>0$, so the third edge must be 'negative enough' to sum to a number ≤ 0 when a number $\geq k$ is added to it. This means $\text{Net}_{\mathscr{U}}(z>x)\leq -k$, which is the same as saying $\text{Net}_{\mathscr{U}}(x>z)\geq k$.

($\Leftarrow$) Assume $\mathscr{U}$ satisfies transfer of strict preference, and let x, y, z be any three alternatives. Without loss of generality, assume that their triangle G is oriented as shown in Fig. 12.

Case 1: *Assume some two edges both have labels* >0. Without loss of generality assume the yx edge label $=j$, the zy edge label $=k$, and $0<k\leq j$. Then $k>0$ and $\text{Net}_{\mathscr{U}}(x>y)\geq k$ and $\text{Net}_{\mathscr{U}}(y>z)\geq k$, so by transfer of strict preference $\text{Net}_{\mathscr{U}}(x>z)\geq k$. Since $l=\text{Net}_{\mathscr{U}}(z>x)=-\text{Net}_{\mathscr{U}}(x>z)$, it follows that $l\leq -k$, so that $l+k\leq 0$. Thus, some pair of edge labels has a sum ≥ 0, and some pair a sum ≤ 0.

Case 2: *Assume some two edges both have labels* ≤ 0. Very similar to Case 1.

Case 3: *Assume one edge has a* 0 *label, one edge a label* >0, *and one edge a label* <0. It is clear in this case that the weak edge sum condition is satisfied.

Case 4: *Assume two or more edges have* 0 *labels.* Again it is clear that the weak edge sum condition is satisfied. □

What is the 'best' version of Sen's Theorem? This all depends on which transitivity condition is viewed as the goal, so there will always be a number of versions around. However, appropriateness arguments can be made for favoring one or more forms of transitivity over others. One such argument states that the appropriate transitivity conditions are those that guarantee a 'Condorcet winner' (see the discussion below). Sen transitivity can be seen to be appropriate in this sense, and it is a straightforward

Observation 12. *If* $\mathscr{U}$ *transfers strict preference, then* $\mathscr{U}$ *is Sen transitive.*

Thus, another reason for interest in Theorems 6 and 12 is that they provide a link with conditions based on the notions of a Condorcet winner. As we will see, weak transitivity is not sufficiently powerful to provide this link.

Definition. Given a profile $\mathscr{U}$, a *Condorcet winner* is an alternative s that at least ties every other alternative in one-on-one matchups with majority rule; equivalently, $sR_{\mathscr{U}}x$ for every alternative x.

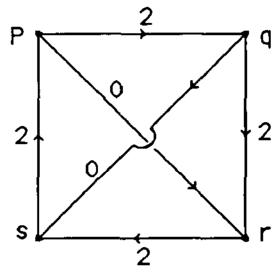


Fig. 13.

Theorem 13. *If $\mathcal{U}$ is Sen transitive, then there exists a Condorcet winner.*

Proof. Construct any $P_{\mathcal{U}}$ chain of maximal length

$$x_1 P_{\mathcal{U}} x_2 P_{\mathcal{U}} x_3 \ldots x_{n-1} P_{\mathcal{U}} x_n.$$

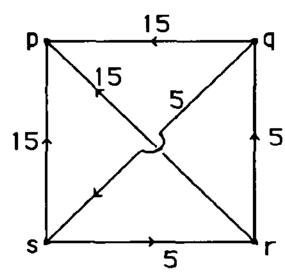
Sen transitivity guarantees that such a maximal chain exists, since it guarantees that no alternative can appear twice in such a $P_{\mathcal{U}}$ chain. It follows that the top element in such a chain (x_1 above) is a Condorcet winner. □

To see that weak transitivity does not suffice, an example will do the trick (Fig. 13). Here are four alternatives. It is true that $\mathcal{U}$ satisfies weak transitivity (there are no length three strong voters' paradoxes), but there is a cycle of majorities among the four (a 'length four voters' paradox'), and thus there is no Condorcet winner.

No transitivity condition could be *equivalent* to the existence of a Condorcet winner, since the presence of one overwhelmingly favored alternative cannot possibly preclude length three voters' paradoxes among the other alternatives (see Fig. 14).

However, the following condition is more comparable to transitivity conditions:

Definition. A profile $\mathcal{U}$ is *hereditarily Condorcet* if for each non-empty subset Y of the set X of alternatives there is some alternative s in Y such that $sR_{\mathcal{U}}y$ for each y in Y (s is a 'local Condorcet winner').



In this example p is a Condorcet winner, yet a strong voters' paradox exists.

Fig. 14.

Theorem 14. *Let $\mathcal{U}$ be any profile. Then*

(a) *$\mathcal{U}$ is hereditarily Condorcet if and only if no strong voters' paradoxes exist (i.e. no cycle of strict majorities of* any *length exists), and*

(b) *$\mathcal{U}$ is Sen transitive $\Rightarrow$ $\mathcal{U}$ is hereditarily Condorcet $\Rightarrow$ $\mathcal{U}$ is weakly transitive.*

Proof. (a) If $\mathcal{U}$ has a voters' paradox of any length, then there can be no local Condorcet winner among the alternatives in this voters' paradox. Hence, if $\mathcal{U}$ is hereditarily Condorcet it has no such paradox.

To say that $\mathcal{U}$ has no strong voters' paradoxes of any length is precisely to say that no alternative can appear twice in any $P_{\mathcal{U}}$ chain. Given this condition and any

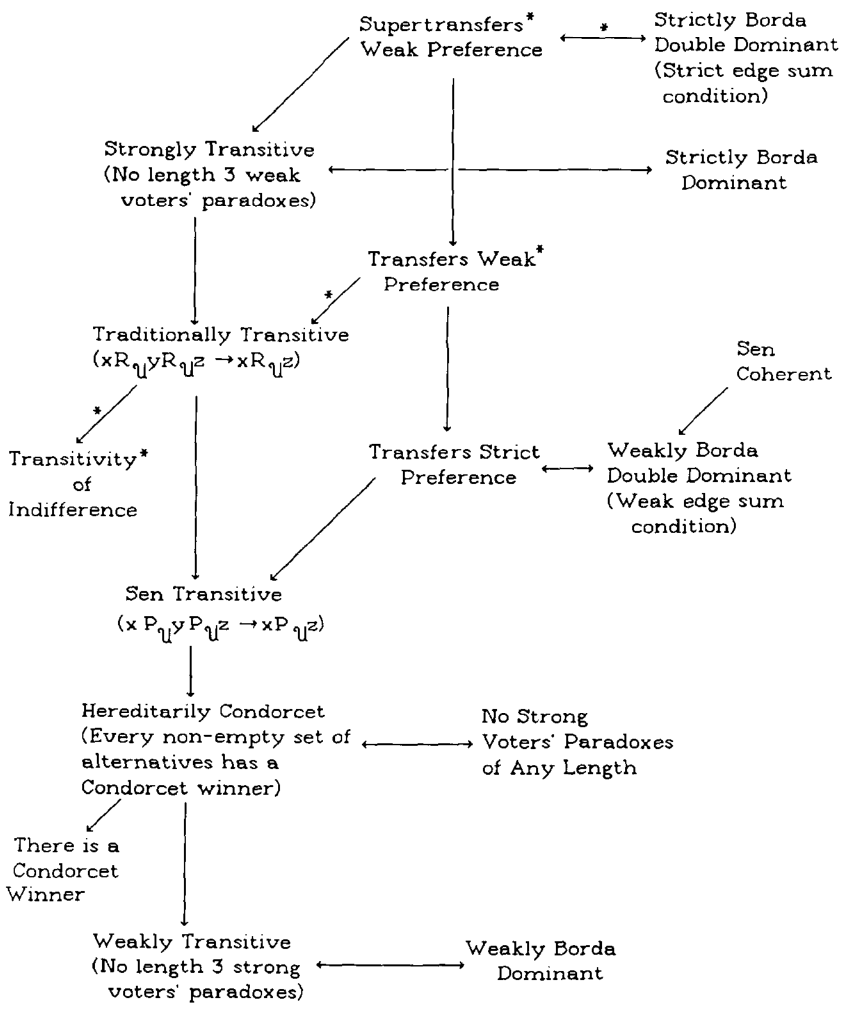


Fig. 15. Summary: parity-free properties of a profile $\mathcal{U}$ (properties and arrows marked with asterisks will be discussed subsequently).

subset Y of alternatives, there must be a $P_{\mathcal{U}}$ chain of maximal length among those chains containing only alternatives in Y. The top alternative in this chain is a local Condorcet winner for the set Y.

(b) If $\mathcal{U}$ is Sen transitive, the argument that it is hereditarily Condorcet is exactly as in the proof of Theorem 13. Since a hereditarily Condorcet profile $\mathcal{U}$ has no strong voters' paradoxes of any length it certainly has no strong voters' paradoxes of length three, so such a $\mathcal{U}$ is weakly transitive. □

We summarize most of the results for profiles $\mathcal{U}$ (of mixed weak and strict preference orders) in Fig. 15. This figure includes all the parity-free properties discussed up to this point, and their logical relationships, except for Corollaries 4 and 5.

It is interesting to note that strict double Borda dominance is also necessary and sufficient for a quantitative form of transitivity.

Definition. *$\mathcal{U}$ supertransfers weak preference* if for every three alternatives x, y, and z and every weakly positive integer $k \geq 0$, if $\text{Net}_{\mathcal{U}}(x>y) \geq k$ and $\text{Net}_{\mathcal{U}}(y>z) \geq k$, then $\text{Net}_{\mathcal{U}}(x>z) > k$. (Note that the last inequality is strict, implying that net preferences grow strictly larger as they transfer.)

Theorem 13. *For any profile $\mathcal{U}$, $\mathcal{U}$ satisfies the strict edge sum condition if and only if $\mathcal{U}$ supertransfers weak preference.*

Proof. ($\rightarrow$) Assume that $\mathcal{U}$ satisfies the strict edge sum condition, i.e. for every three alternatives x, y, and z and every cycling orientation of the corresponding triangle G, $G_{\mathcal{U}}$ has two edge labels with a sum >0, and two with a sum <0. Let $k \geq 0$ with $\text{Net}_{\mathcal{U}}(x \geq y) \geq k$ and $\text{Net}_{\mathcal{U}}(y>z) \geq k$. Orient G as in Fig. 16 and observe that two edges receive labels $\geq k$. Since the sum of these labels is $\geq 2k \geq 0$, the third must receive a label negative enough to sum to a number <0 when a number $\geq k$ is added to it. This means that $\text{Net}_{\mathcal{U}}(z>x) < -k$, which means that $\text{Net}_{\mathcal{U}}(x>z) > k$, as desired.

($\leftarrow$) Assume that $\mathcal{U}$ satisfies supertransfer of weak preference, and let x, y, and z be any three alternatives. Without loss of generality, assume that their triangle G is oriented as in Fig. 17.

Case 1: *Some two edges both have labels* ≥ 0. Without loss of generality assume $0 \leq k \leq j$. Then $k \geq 0$ and $\text{Net}_{\mathcal{U}}(x>y) \geq k$ and $\text{Net}_{\mathcal{U}}(y>z) \geq k$, so by our assump-

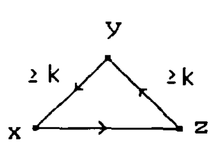


Fig. 16.

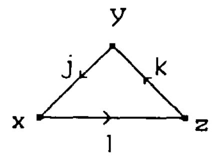


Fig. 17.

tion $\text{Net}_{\mathcal{U}}(x>z)>k$, so $l=\text{Net}_{\mathcal{U}}(z>x)<-k\leq 0$. Then $j+k\geq 0$, while $k+l<0$. It remains only to show that $j+k\neq 0$.

But if $j+k=0$, then $j=k=0$. In this situation, we would have $\text{Net}_{\mathcal{U}}(z>y)\geq 0$ and $\text{Net}_{\mathcal{U}}(y>x)\geq 0$, and conclude $\text{Net}_{\mathcal{U}}(z>x)>0$, i.e. $l>0$. This is a contradiction since we have already shown that $l<0$, so under the assumption of super-transfer of weak preference, $j+k=0$ is impossible.

Case 2: *Some two edges both have labels* ≤ 0—similar. □

Now let us define: $\mathcal{U}$ *transfers weak preference* if for every three alternatives x, y, and z and every weakly positive integer k, if $\text{Net}_{\mathcal{U}}(x>y)\geq k$ and $\text{Net}_{\mathcal{U}}(y>z)\geq k$, then $\text{Net}_{\mathcal{U}}(x>z)\geq k$. Then it is an easy

Observation 14. *If* $\mathcal{U}$ *transfers weak preference, then* $\mathcal{U}$ *is traditionally transitive.*

We end this section with a brief discussion of the transitivity of indifference. Given alternatives x and y we will say that a profile $\mathcal{U}$ is *indifferent* to x and y (notation: $xI_{\mathcal{U}}y$) if both $xR_{\mathcal{U}}y$ and $yR_{\mathcal{U}}x$. Equivalently, $xI_{\mathcal{U}}y$ if and only if $\text{Net}_{\mathcal{U}}(x>y)=0=\text{Net}_{\mathcal{U}}(y>x)$.

$\mathcal{U}$ satisfies *transitivity of indifference* if for each three alternatives x, y, and z, if $xI_{\mathcal{U}}y$ and $yI_{\mathcal{U}}z$, then $xI_{\mathcal{U}}z$. It is easy to see that if $\mathcal{U}$ is traditionally transitive, then $\mathcal{U}$ satisfies transitivity of indifference, since if $xI_{\mathcal{U}}yI_{\mathcal{U}}z$, then both $xR_{\mathcal{U}}yR_{\mathcal{U}}z$ and also $zR_{\mathcal{U}}yR_{\mathcal{U}}x$ from which traditional transitivity implies both $xR_{\mathcal{U}}z$ and $zR_{\mathcal{U}}x$. Hence:

Observation 15. *If* $\mathcal{U}$ *is traditionally transitive, then* $\mathcal{U}$ *satisfies transitivity of indifference.*

This completes our discussion of Fig. 15.

11. Parity considerations

When assumptions are made about the number of concerned individuals (that it is an odd number, for example) hypotheses such as Sen coherence or weak Borda dominance yield stronger conclusions than they otherwise would. We will consider two parity conditions:

Parity condition 1. For every pair, x and y, of distinct alternatives the number of concerned individuals (individuals not indifferent to x and y) is odd.

Parity condition 2. For every triple x, y, and z of distinct alternatives the number of concerned individuals (individuals who do not rank all three at the same level) is odd.

Observe that to be concerned among three alternatives means to be concerned with at least one pair from the three.

Neither of the two conditions implies the other (see Corollary 18) so strictly speaking neither is stronger. In terms of consequences, however, the first has a more far-reaching effect.

Sen's Theorem (Original version). *In the presence of Parity condition* 2, *if $\mathcal{U}$ is a Sen coherent profile, then $\mathcal{U}$ is traditionally transitive.*

The proof is omitted. See Sen (1966) and Taylor (unpublished).

Part of the importance of this theorem lies in the particular way in which the two assumptions reinforce each other. Other conditions considered in this paper do *not* appear to be greatly boosted by Parity condition 2. In particular,

Observation 16. *Even in the presence of Parity condition* 2, *weak Borda double dominance does not imply traditional transitivity.*

Proof. There is a simple counter-example. The profile $\mathcal{U}$, consisting of three individuals whose preference orders are

$$\begin{bmatrix} r \\ t \\ s \end{bmatrix}, \quad \begin{bmatrix} s \\ rt \end{bmatrix}, \quad \begin{bmatrix} st \\ r \end{bmatrix} \qquad \text{(that is, } \vec{u},\ \vec{\beta}, \text{ and } \vec{\gamma}\text{)},$$

is weakly Borda double dominant and clearly satisfies the Parity condition 2. Note that $rR_{\mathcal{U}}t$ and $tR_{\mathcal{U}}s$ but it is not the case that $rR_{\mathcal{U}}s$.

Comment. In some sense, the equivalence between weak Borda double dominance and transfer of strict preference represents a strictly better theorem than Sen's, since the assumption of weak Borda double dominance is weaker (applies to more profiles) than Sen coherence, the conclusion of transfer of strict preference is stronger than Sen transitivity, and the implication is reversible.

However, Parity condition 2 combines with Sen coherence in a surprisingly synergistic way—thus no result in this paper displaces the original parity-dependent form of Sen's Theorem.

The effect of Parity condition 1 is more straightforward; clearly $\text{Net}_{\mathcal{U}}(x>y)=0$

becomes impossible, since this would mean the number strictly preferring x to y would equal the number strictly preferring y to x, making the number concerned an even integer. This means that for no triangle G can any of $G_{\mathscr{U}}$'s edge labels be 0, so that $xR_{\mathscr{U}}y$ becomes synonymous with $xP_{\mathscr{U}}y$.

An immediate consequence is that all five[1] qualitative transitivity conditions become equivalent, so that (for example) the weakest dominance assumption, weak Borda dominance, implies the strongest qualitative transitivity (strong transitivity) in the presence of the Parity condition 1. Similarly, transfer of weak and strict preference become equivalent in the presence of the Parity condition 1.

If $\mathscr{U}$ is a profile of strict preference orders only, then it is immediate that the two parity conditions become equivalent to each other, and to 'the number of individuals is odd'. It is straightforward to trace the strengthening effect of parity in the case of a profile of strict preference orders.

By translating the two parity conditions into vote count terms, it becomes possible to establish the earlier claim that neither parity condition implies the other. It helps to fix attention on a given triple s, t, r of alternatives.

Lemma 17. *Let $\mathscr{U}$ be a profile, s, t, and r be three distinct alternatives and Vote count ($\mathscr{U}$) for s, t, r be $(a, b, c, d, e, f, g, h, i, j, k, l)$. Then*

(a) *The number of individuals concerned with each pair of alternatives from s, t, and r is odd if and only if both*

(i) *$a+b+c+d+e+f$ is odd, and*

(ii) *$g+h$, $i+j$, and $k+l$ have the same parity (i.e. each of the three is even, or each is odd).*

(b) *The number of individuals concerned with s, t, and r (i.e. concerned with at least one pair of these alternatives) is odd if and only if*

(iii) *$a+b+c+d+e+f+g+h+i+j+k+l$ is odd.*

Since it is straightforward to construct an example of a 12 tuple satisfying (i) and (ii) but not (iii) (i.e. by having each pair sum in (ii) be odd), and another satisfying (iii) but not (i) (by having $a+b+c+d+e+f$ be even and $g+h+i+j+k+l$ be odd), we get

Corollary 18. *Neither of the two parity conditions implies the other (specifically, this is so when there are exactly three alternatives).*

Proof of Lemma 17. Note that (b) is immediate, since $a+b+c+d+e+f+g+h+i+j+k+l$ *is* the number of individuals concerned with at least one pair from among s, t, and r.

[1] Weak, Sen, Traditional, Strong transitivity, and Hereditarily Condorcet.

For (a), let

$$Q1 = a+b+c+d+e+f+g+h+i+j,$$

$$Q2 = a+b+c+d+e+f+i+j+k+l,$$

and

$$Q3 = a+b+c+d+e+f+g+h+k+l.$$

Observe that these three represent the number of individuals concerned with r and s, with r and t, and with s and t, respectively. Thus, we wish to establish that Q1, Q2, and Q3 are each odd, if and only if (i) and (ii) both hold.

($\rightarrow$) If Q1, Q2, and Q3 are each odd, then so is their sum, $Q1+Q2+Q3 = 3[a+b+c+d+e+f]+2[g+h+i+j+k+l]$, from which it follows that $a+b+c+d+e+f$ is odd. This, together with Q1's oddness, implies that $g+h+i+j$ is even. Similarly, $i+j+k+l$ and $g+h+k+l$ are each even. Thus, we know about the three quantities $g+h$, $i+j$, and $k+l$ that the sum of any two of them is even. It is easy to see that this happens precisely when all three have the same parity.

($\leftarrow$) It is straightforward to check that if (i) and (ii) both hold, then Q1, Q2, and Q3 are all odd. □

Appendix: Mathematical comments and new directions

Mathematical comment 1. These characterizations depend upon the assumption that G is a triangle, oriented in one standard way. The more general characterization, for an arbitrary directed graph, is that in the cyclic component the net flux into each node is zero, while in the cocyclic component the net flux around every loop is zero. This means that the cyclic component can be represented as a linear combination of unitary loop flows (Fig. 18). The cocyclic component can be represented as a linear combination of unitary sinks and sources (Fig. 19). These more general representations are non-unique when there are four or more nodes in the (complete) graph G.

Mathematical comment 2. Actually, for any profile $\mathscr{U}$, $G_{\mathscr{U}}$ may be considered to be a vector in the space $V_1[G]$. Literally, $V_1[G]$ is a free $\mathbb{Z}$-module, but it can safely be thought of as the vector space the basis of which is the set of edges of G.

The boundary map ∂ of homology theory is a linear transform from $V_1[G]$ to the

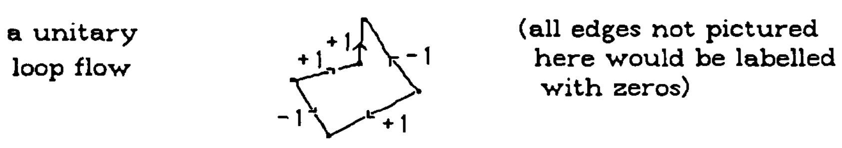


Fig. 18.

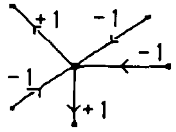


Fig. 19.

space $V_0[G]$, which can be thought of as the vector space the basis of which is the set of vertices of G. The kernel of this map is a subspace of $V_1[G]$ known as the cycle subspace, $Z_1[G]$; this is discussed in Croom (1978) and Harary (1958). Any vector $\vec{v}$ in $V_1[G]$ can be uniquely decomposed into a component in $Z_1[G]$ and a component orthogonal to $Z_1[G]$ (there is a natural inner product on these spaces). The 'fundamental decomposition' is precisely this decomposition, applied to $G_{\mathcal{U}}$. It follows immediately, then, that adding several vectors, then decomposing, has the same effect as decomposing before adding. It also follows that the fundamental decomposition is unique.

In the special case that G is a triangle (whether because there are only three alternatives or because we are focusing on alternatives three at a time, as throughout this paper), $Z_1[G]$ is one dimensional, which is why the cyclic component can be represented by a single scalar, the 'spin'. It follows that the complementary subspace must be two dimensional, and isomorphic to a two-dimensional subspace of $V_0[G]$ (which is itself three-dimensional). See MC 4.

Mathematical comment 3. Grzegorz Lissowski has commented that the fractional quantities that appear in the decomposition of an individual preference order are unintuitive. Perhaps this is so when one views $G_{\mathcal{U}}$ solely as a labelled graph in which the edge labels represent information as to which of the endpoint vertices is preferred over the other; after all, one either prefers or does not and fractional preference is not in the picture for the voting systems we have in mind. When one realizes that the quantities are actually vectors, however, it becomes quite natural for a vector the components of which are entirely integer in one basis, to have a decomposition into vectors having non-integral components. We could fix the 'problem' of fractional values in the cyclic and cocyclic components by tripling all edge labels in the definition of $G_{\mathcal{U}}$, but this does not seem natural.

Mathematical comment 4. Under the appropriate choice of basis, $\partial(G_{\mathcal{U}}) = 2(s^{\mathcal{B}}, t^{\mathcal{B}}, r^{\mathcal{B}})$. These Borda counts always sum to zero, which is why the image of ∂ is actually a two-dimensional subspace of the three-dimensional space $V_0[G]$.

Actually our choice of these weights (and of the fractional weights introduced for weak preference orders in Section 9) is derived from the boundary map ∂—the weights are chosen to make $\partial(G_{\mathcal{U}})$ proportional to $(s^{\mathcal{B}}, t^{\mathcal{B}}, r^{\mathcal{B}})$.

Directions for new research

Consider the map $\mathcal{U} \mapsto G_{\mathcal{U}}$ that assigns to each profile the corresponding labelled directed graph. As suggested by MC 2, this is actually a map from $V_1[G]^{|N|}$ to $V_1[G]$, where $|N|$ is the number of individuals. This map, which is easily seen to be linear, will be denoted by χ_1 and we will use the standard symbol ∂ for the boundary map of homology. Let us define a social welfare function C to be *one dimensional* if it factors through χ_1—that is, $C = C^* \circ \chi_1$.

Observation 19. *For any profile $\mathcal{U}$, $(\partial \circ \chi_1)(\mathcal{U})$ is the vector the x component of which, for any alternative x, is just $x^{\mathcal{B}}$ (x's Borda count for the profile $\mathcal{U}$) scaled by a factor determined by the number of alternatives. Thus, the Borda count is one dimensional and any other one-dimensional social welfare function that ignores $G_{\mathcal{U}}$'s cyclical component and is 'linear' must, in effect, be the Borda count (more precisely, any linear map with domain $V_1[G]$ and kernel containing $Z_1[G]$ must factor through ∂).*

A number of interesting questions are suggested by this observation. Which social welfare functions fail to be one dimensional, and how do they use the information not in $\chi_1(\mathcal{U})$? Is there a useful description of χ_1's 'kernel' (or of the part of the kernel these functions actually use)? How do the one-dimensional social welfare functions use the cyclical component of $G_{\mathcal{U}}$ and/or fail to be linear? An important second example of a one-dimensional social welfare function is 'take the transitive closure of $R_{\mathcal{U}}$'; this is the same as peeling off Condorcet winners one at a time from the top, once those alternatives locked in cycles have been declared tied.

One fairness principle, with many supporters, is that a good social welfare function should choose the Condorcet winner whenever she exists. Indeed, one of Condorcet's criticisms of the Borda count is that it sometimes fails this principle. Apparently, the phenomenon arises from masked cycles, in certain profiles, that shift the Condorcet winner away from the Borda winner. The decomposition of such profiles suggests there may exist some new counter-arguments to Condorcet's criticism, and we hope to strengthen such counterarguments via some precise theorems.

Empirical election data provide a third direction of exploration. We intend to apply the fundamental decomposition to real election returns to see what crops up, and also to determine the relative frequency of Borda double dominance ('quantitative' transitivity) and Borda dominance ('qualitative' transitivity).

Acknowledgments

The author would like to thank Steven Brams and Alan Taylor for helpful comments, and for introducing me to Social Choice Theory. Thanks also to Grzegorz

Lissowski for comments on an earlier draft that led to several changes.

This research was partially supported by NSF Grant DMS-890095, and by the Sloan Foundation.

References

D. Black, The Theory of Committees and Elections (Cambridge University Press, Cambridge, 1958).

S. Brams, Rational Politics (CQ Press, Washington, 1985).

F. Croom, Basic Concepts of Algebraic Topology (Springer-Verlag, New York, 1978).

F. Harary, Graph theory and electric networks, IRE Trans. Circuit Theory CT-6 (May 1958) 95–109.

A.K. Sen, A possibility theorem on majority decisions, Econometrica 34 (2) (April 1966) 491–499.

A. Taylor, Discrete and Deductive Mathematics for Political Science (unpublished book manuscript).

W. Zwicker, Homological Foundations of Social Choice Theory (in preparation).