Let $V$ be a fin dim vector space over a field $F$.

<u>Defn</u> A <u>linear operator</u> on $V$ (or <u>endomorphism</u> of $V$) is a linear transformation $V \to V$.

<u>Notation</u> · $\mathcal{L}(V) := \mathcal{L}(V,V)$.

· If $f \in \mathcal{L}(V)$ and $\alpha$ is an ordered basis of $V$, $M_\alpha(f) = M_\alpha^\alpha(f)$ denotes the matrix of $f$ wrt $\alpha$.

<u>Goal</u> Given $f \in \mathcal{L}(V)$ find a basis $\alpha$ for $V$ s.t. $M_\alpha(f)$ is especially simple.

Suppose, for instance, that $\alpha = \{v_1, \dots, v_n\}$ is a basis for $V$ s.t. $M_\alpha(f) = \mathrm{diag}(c_1, \dots, c_n)$. Then (HW):

· A basis for $\mathrm{im}(f)$ is $\{v_i \mid c_i \neq 0\}$ and $\mathrm{rank}(f) = |\{i \mid c_i \neq 0\}|$

· A basis for $\ker(f)$ is $\{v_i \mid c_i = 0\}$ and $\mathrm{null}(f) = |\{i \mid c_i = 0\}|$.

· $\det(f) = c_1 \cdots c_n$.

We'll address the following:

· Which linear operators on $V$ can be represented by a diagonal matrix?

· If not diagonal, what is the simplest type of matrix by which we can represent a given operator?

<u>Defn</u> A scalar $\lambda \in F$ is an <u>eigenvalue</u> of $f$ if $\exists$ nonzero $v \in V$ s.t. $f(v) = \lambda v$. In that case, $v$ is an <u>eigenvector</u> of $f$ with eigenvalue $\lambda$.

<u>e.g.</u> $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \in \mathcal{L}(\mathbb{R}^2)$ Then $f(2,3) = (4,6) = 2\cdot(2,3)$,
$(x,y) \longmapsto (2y - x, 6y - 6x)$

so $v_1 = (2,3)$ is an eigenvector of $f$ with eigenvalue 2. Similarly, $f(1,2) = (3,6) = 3\cdot(1,2)$, so $v_2 = (1,2)$ is an eigenvector of $f$ with eigenvalue 3.

Consider $\alpha = \{v_1, v_2\}$. Then $M_\alpha(f) = \mathrm{diag}(2,3) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Prop Let $f \in \mathcal{L}(V)$ and suppose $\alpha = \{v_1, \dots, v_n\}$ is a basis for $V$ consisting of eigenvectors of $f$. Then $M_\alpha(f) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ where $\lambda_i$ = eigenvalue of $v_i$ for $f$.

Pf Follows from defn of $M_\alpha(f)$. $\square$

Defn A linear operator $f \in \mathcal{L}(V)$ is diagonalizable if $\exists$ basis of $V$ consisting of eigenvectors of $f$.

e.g. $f \in \mathcal{L}(\mathbb{R}^2)$ given by $f(x,y) = (-y, x)$. Claim $f$ has no eigenvector. If $(a,b)$ is an eigenvector of $f$ w/ eigenvalue $\lambda$, then $(-b, a) = f(a,b) = \lambda(a,b)$ so $-b = \lambda a$, $a = \lambda b$

$$\Rightarrow \quad b(\lambda^2+1) = a(\lambda^2+1) = 0.$$

Since $(a,b) \neq (0,0)$, get $\lambda^2 + 1 = 0$ ⚡ for $\lambda \in \mathbb{R}$.

Defn Let $f \in \mathcal{L}(V)$. The characteristic polynomial of $f$ is the polynomial $p_f(x) \in F[x]$ given by $p_f(x) = \det(A - xI)$ where $A = M_\alpha(f)$ for any ordered basis $\alpha$ of $V$.

Prop Suppose $\alpha, \beta$ ordered bases of $V$, let $A = M_\alpha(f)$, $B = M_\beta(f)$. Then $\det(A - xI) = \det(B - xI)$.

Pf $\exists$ invertible $P$ s.t. $A = P^{-1}BP$. Thus $P^{-1}(B - xI)P = P^{-1}BP - P^{-1}xIP$
$= A - xI$ so $B - xI$, $A - xI$ are similar. Finally,

$$\begin{aligned} \det(A - xI) &= \det(P^{-1}(B - xI)P) \\ &= \det P^{-1} \det(B - xI) \det P \\ &= \det(B - xI) \quad \left(\text{b/c } \det P^{-1} = \frac{1}{\det P}\right). \ \square \end{aligned}$$

e.g. For $f(x,y) = (-y, x)$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$p_f(x) = \det\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right) = \det\begin{pmatrix} -x & -1 \\ 1 & -x \end{pmatrix} = x^2 + 1.$$

<u>Lemma</u> Let $\varphi \in \mathcal{L}(V)$. Then $\varphi$ is invertible iff $\ker \varphi = \{0\}$.

<u>Pf</u> Rank-nullity. $\square$

<u>Prop</u> Let $f \in \mathcal{L}(V)$, $\lambda \in F$. Then

$$\lambda \text{ is an eigenvalue of } f \iff \lambda \text{ is a root of } p_f(x).$$

<u>Pf</u> Let $\varphi = f - \lambda I \in \mathcal{L}(V)$. Then $\lambda$ is an eigenvalue of $f$ iff $\ker \varphi \neq \{0\}$ iff $\varphi$ not invertible iff $p_f(\lambda) = 0$. $\square$

<u>e.g.</u> Let $F = \mathbb{Z}/11\mathbb{Z}$, $f \in \mathcal{L}(F^2)$ given by $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.

Then $p_f(x) = \det(A - xI) = \det\begin{pmatrix} 2-x & 1 \\ 1 & 3-x \end{pmatrix} = x^2 - 5x + 5 = (x-6)(x-10)$

so the eigenvalues of $f$ are 6 and 10.