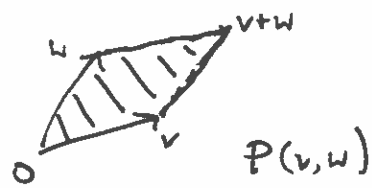
MATH 201: LINEAR ALGEBRA
DETERMINANTS OVER $\mathbb{R}$

Let $v = (x_1, y_1), w = (x_2, y_2) \in \mathbb{R}^2$ be linearly independent vectors. They span the parellelogram

$$P(v, w) = \{av + bw \mid 0 \le a, b \le 1\}.$$

Problem 1. Let M be the matrix with columns v, w so that $M = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$. Show that $M([0,1]^2) = P(v, w)$ and draw a picture of $P(v, w)$. (Here $[0,1]^2 = [0,1] \times [0,1] = \{(a, b) \mid 0 \le a, b \le 1\}$.)

$M\begin{pmatrix} a \\ b \end{pmatrix} = av + bw$ so $M([0,1]^2) = P(v,w)$.



If the vectors v, w are linearly dependent, it is reasonable to say that the degenerate parallelogram $P(v, w)$ has area 0. This defines a function

$$A : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

given by $A(v, w) = \text{area}(P(v, w))$.

Problem 2. Let $k \in \mathbb{R}$. What is $A(kv, w)$? (Be careful with the case $k < 0$.)

For $k \geq 0$, $A(kv,w) = k\,A(v,w)$. In general,

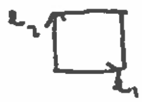
$$A(kv,w) = |k|\,A(v,w).$$

(Multiplying v by k scales the base of the parallelogram by $|k|$.)

Problem 3. Let $k \in \mathbb{R}$. What is $A(v, w + kv)$? (A proof by picture might be appropriate.)

Rotate so that v is on the horizontal axis. Then $P(v,w)$ and $P(v,w+kv)$ have the same base and height. Thus $A(v,w+kv) = A(v,w)$.

Problem 4. What is $A(e_1, e_2)$?



$A(e_1, e_2) = 1.$

Problem 5. The function A nearly has the properties of a determinant function. Explain what proprties it does and does not have in this respect.

The function A is alternating and normalized, but not quite multilinear as scalars pull out as their absolute value.

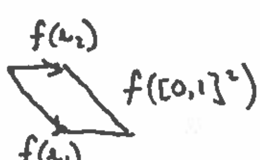
This inspires us to define the *signed area* of $P(v, w)$. For this definition, the order of v and w matters. If v and w are linearly independent, let θ be the angle from v to w, measured counter-clockwise. Then $0 < \theta < 2\pi$ and $\theta \neq \pi$. We can then define

$$SA(v,w) = \begin{cases} A(v,w) & \text{if } 0 < \theta < \pi, \\ -A(v,w) & \text{if } \pi < \theta < 2\pi, \\ 0 & \text{if } v, w \text{ linearly dependent.} \end{cases}$$

Problem 6. Prove that $SA(v, w) = \det M$ where M has columns v, w.

SA is alternating, multilinear, and normalized as a function of the columns of a 2×2 matrix. By our $\det M^T = \det M$ theorem, this is equivalent to $SA(v,w) = \det \begin{pmatrix} \vdots & \vdots \\ v & w \\ \vdots & \vdots \end{pmatrix}$.

Problem 7. Consider the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$. Draw a picture of $f([0,1]^2)$. What is its area?



$\det \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} = 0 - (-2) = 2 = \text{area}(f([0,1]^2))$.

Now let $(v_1, \ldots, v_n)$ be an n-tuple of vectors in $\mathbb{R}^n$ (*i.e.*, an element of $(\mathbb{R}^n)^n$). The *parallelepiped* formed by $(v_1, \ldots, v_n)$ is the set

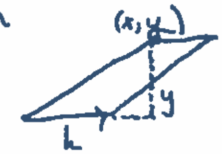
$$P(v_1, \ldots, v_n) = \{t_1 v_1 + \cdots + t_n v_n \mid t_1, \ldots, t_n \in [0,1]\}.$$

When $n = 2$, this gives the parallelogram $P(v_1, v_2)$. For $n = 3$, we get a solid prism as long as the vectors are linearly independent.

We define the *volume* of a parallelepiped determined by $(v_1, \ldots, v_n)$ as the absolute value of the determinant of the $n \times n$ matrix with columns $v_1, \ldots, v_n$.

Problem 8. Using the properties of the determinant and your intution about how a volume should behave, argue why this definition makes sense. Check it against standard formulas for area and volume when $n = 2$ and $n = 3$.

Here is the $n=2$ check: If $v=(k,0)$, $w=(x,y)$, then $\det\begin{pmatrix} k & x \\ 0 & y \end{pmatrix} = ky - x\cdot 0 = ky$. Geometrically, $P(v,w)$ is



with area xy. For the general case, consider the rotation $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ that takes v to the positive x-axis. Rotations don't change area, and $\det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \cos^2\theta + \sin^2\theta = 1$

so area is preserved.

Problem 9. For $n \times n$ real matrices A, B, interpret the rule $\det(AB) = \det(A)\det(B)$ in terms of volumes.

Since absolute value commutes with products as well, we see that the volume of $(AB)([0,1]^n)$ is the product of the volume of $A[0,1]^n$ and $B[0,1]^n$.