V fin dim vs $/F$, $f \in \mathcal{L}(V)$. When is f diagonalizable?

Defn For $U_1, \ldots, U_k$ subspaces of V, say V is the direct sum of $U_1, \ldots, U_k$ written $V = U_1 \oplus \cdots \oplus U_k$ if $\forall v \in V$, $\exists! u_i \in U_i$ s.t. $v = u_1 + \cdots + u_k$.

Prop Suppose $V = U_1 \oplus \cdots \oplus U_k$ and let B_i be a basis for U_i. Then

(a) $B_i \cap B_j = \emptyset$ for $i \neq j$,

(b) $B = B_1 \cup \cdots \cup B_k$ is a basis for V,

(c) $\dim V = \dim U_1 + \cdots + \dim U_k$.

Pf HW. $\square$

Lemma Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of $f \in \mathcal{L}(V)$, and suppose u_i is a λ_i eigenvector of f. Then $u_1, \ldots, u_k$ are lin ind.

Pf by induction on k. If $k=1$, $u_1 \neq 0$ so $\{u_1\}$ lin ind. Assume now $k>1$ and the result holds for $k-1$. If $c_1u_1 + \cdots + c_ku_k = 0$ for $c_i \in F$, apply $f - \lambda I_k$ to both sides:

$$0 = \sum_{i=1}^{k} (c_i\lambda_i u_i - \lambda_k c_i u_i) = \sum_{i=1}^{k-1} c_i\lambda_i u_i - \lambda_k c_i u_i$$

$$= \sum_{i=1}^{k-1} (c_i\lambda_i - \lambda_k c_i) u_i .$$

By ind'n hypothesis, $c_i\lambda_i - \lambda_k c_i = 0$ for $i<k$.

$$\Rightarrow c_i(\lambda_i - \lambda_k) = 0$$

Since $\lambda_i \neq \lambda_k$, get $c_i = 0$ for $i<k$ so

$$0 = c_1u_1 + \cdots + c_ku_k = c_ku_k$$

$\Rightarrow c_k = 0$ too. $\square$

Defn For $\lambda \in F$, the λ-eigenspace of f is the subspace

$$E_\lambda(f) = \ker(f - \lambda I)$$
$$= \{\lambda\text{-eigenvectors of } f\} \cup \{0\}.$$

Lemma Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of f, and let $U = E_{\lambda_1}(f) + \cdots + E_{\lambda_k}(f)$. Then $U = E_{\lambda_1}(f) \oplus \cdots \oplus E_{\lambda_k}(f)$.

Pf Suffices to show $u_1 + \cdots + u_k = 0$, $u_i \in E_{\lambda_i}(f) \implies u_i = 0 \ \forall i$.
Let $\{u_{i_1}, \dots, u_{i_m}\}$ be nonzero u_i's. Then

$$u_{i_1} + \cdots + u_{i_m} = 0 \text{ is a trivial lin combo of}$$

lin ind vectors $\Rightarrow\Leftarrow$. $\square$

Thm Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of f, and let $d_i = \dim E_{\lambda_i}(f)$. TFAE:

(a) f is diagonalizable

(b) $V = E_{\lambda_1}(f) \oplus \cdots \oplus E_{\lambda_k}(f)$

(c) $\dim V = d_1 + \cdots + d_k$.

Pf Let $U_i = E_{\lambda_i}(f)$, $U = U_1 \oplus \cdots \oplus U_k$.

(a) $\Rightarrow$ (b): If f is diagonalizable then every elt of V is a linear combo of eigenvectors of f. Since every eigenvector is in U_i for some i, get $V = U$.

(b) $\Rightarrow$ (c): ✓

(c) $\Rightarrow$ (a): For $i = 1, \dots, k$, let B_i be a basis of U_i. Then $B = B_1 \cup \cdots \cup B_k$ is a basis of U with $d_1 + \cdots + d_k$ elts so $\dim V = \dim U$. As $U \subseteq V$, get $U = V$. So B is a basis of V, and every elt of B is an eigenvector of f. $\square$

How do we determine $d_1, \dots, d_k$? Choosing a basis of V gives $V \xrightarrow{\cong} F^n$, $\mathcal{L}(V) \xrightarrow{\cong} M_{n \times n}(F)$

$$f \longmapsto A$$

and $\ker(f - \lambda I) \longleftrightarrow \ker(A - \lambda I)$. So d_i can be computed by reducing $A - \lambda I$, counting non-pivot columns e.g. $f \in \mathcal{L}(\mathbb{R}^2)$, $f(x,y) = (-x+2y, -6x+6y)$

$$\updownarrow e_1, e_2$$

$$A = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$$

$$p_f(x) = \det(A - xI) = \det\begin{pmatrix} -1-x & 2 \\ -6 & 6-x \end{pmatrix} = x^2 - 5x + 6 = (x-2)(x-3)$$

$\Longrightarrow$ eigenvalues 2, 3.

$A - 2I \rightsquigarrow \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix}$ $\qquad$ $A - 3I \rightsquigarrow \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$

so $\ker(A-2I)$, $\ker(A-3I)$ are both 1-dim'l, spanned by $(2,3)$, $(1,2)$, resp. Since $1+1=2=\dim \mathbb{R}^2$, f is diagonalizable.

$$M_{\langle (2,3),(1,2) \rangle}(f) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$