Permutation Expansion of the Determinant

Defn A <u>permutation</u> of a set X is a bijective fn $X \to X$. The set of all permutations of X is called the <u>symmetric group</u> on X.

The symmetric group on $\underline{n} = \{1, \dots, n\}$ is the <u>symmetric group on n letters</u>, denoted Σ_n (or S_n, or $\mathfrak{S}_n$).

Represent $\sigma \in \Sigma_n$ by the $2 \times n$ matrix

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \underbrace{\sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n)}_{\text{rearrangement of } 1, \dots, n} \end{pmatrix}$$

Get $|\Sigma_n| = n!$

<u>e.g.</u> The 6 elements of Σ_3 are

$$\begin{pmatrix} 1&2&3\\1&2&3 \end{pmatrix}, \begin{pmatrix} 1&2&3\\2&1&3 \end{pmatrix}, \begin{pmatrix} 1&2&3\\3&2&1 \end{pmatrix}, \begin{pmatrix} 1&2&3\\1&3&2 \end{pmatrix}, \begin{pmatrix} 1&2&3\\2&3&1 \end{pmatrix}, \begin{pmatrix} 1&2&3\\3&1&2 \end{pmatrix}$$

Defn The <u>permutation matrix</u> corresponding to $\sigma \in \Sigma_n$ is the matrix

$$P_\sigma \in M_{n \times n}(F)$$

with i-th column $e_{\sigma(i)}$.

<u>e.g.</u> $\sigma = \begin{pmatrix} 1&2&3\\2&3&1 \end{pmatrix} \rightsquigarrow P_\sigma = \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$

<u>Prop</u> (a) The $\sigma(i)$-th row of P_σ is e_i.

(b) $P_\sigma P_\sigma^T = I_n$ (P_σ is orthogonal)

(c) $\{P_\sigma \mid \sigma \in \Sigma_n\}$ is the set of matrices in $M_{n\times n}(F)$ with exactly one 1 in every row & column, 0's elsewhere.

(d) $P_{\sigma \circ \tau} = P_\sigma P_\tau \quad \forall \sigma, \tau \in \Sigma_n$.

<u>Pf</u> (a) The cols of P_σ are $e_{\sigma(1)}, \dots, e_{\sigma(n)}$. If $j = \sigma(i)$, the j-th row of P_σ is $(e_{\sigma(1)j}, \dots, e_{\sigma(i)j}, \dots, e_{\sigma(n)j}) = e_i$.

(b) $(P_\sigma)_{ab} = e_{\sigma(b)a} = \delta_{\sigma(b),a}$. Thus

$$(P_\sigma P_\sigma^T)_{ij} = \sum_{k=1}^n (P_\sigma)_{ik} (P_\sigma^T)_{kj} = \sum_{k=1}^n (P_\sigma)_{ik} (P_\sigma)_{jk}$$

$$= \sum_{k=1}^n \delta_{\sigma(k) i} \delta_{\sigma(k) j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$\Rightarrow P_\sigma P_\sigma^T = I_n$.

(c), (d): Moral exc. $\square$

<u>Rmk</u> (1) If $AA^T = I_n$ then $1 = \det A \det A^T = (\det A)^2$ so $\det A = \pm 1$. Thus $\det P_\sigma = \pm 1$ $\forall \sigma \in \Sigma_n$.

(2) Fun to think about permutation matrices as "non-attacking rooks" on an $n \times n$ chessboard.

<u>Defn</u> A <u>transposition</u> in Σ_n is a permutation which interchanges two elts of $\underline{n}$ and fixes all others. Write $(a\,b)$ for the transposition swapping a, b.

<u>Defn</u> The <u>sign</u> of a permutation $\sigma \in \Sigma_n$ is $\mathrm{sgn}(\sigma) = \det(P_\sigma) \in \{\pm 1\}$.

<u>Prop</u> Suppose σ is the composition of k permutations. Then $\mathrm{sgn}(\sigma) = (-1)^k$.

<u>Pf</u> If $k = 1$, P_σ obtained from a single row swap so $\det P_\sigma = -1$.
If $\sigma = \tau_1 \circ \cdots \circ \tau_k$ for $k > 1$, τ_i transpositions, then ~~st~~ $P_\sigma = P_{\tau_1} \cdots P_{\tau_k}$ and $\det P_\sigma = \det P_{\tau_1} \cdots \det P_{\tau_k} = (-1)^k$. $\square$

<u>Rmk</u> In Math 332 you'll prove that every elt of Σ_n is a composition of transpositions.

<u>Thm</u> For every $A \in M_{n \times n}(F)$,

$$\det A = \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}$$

Pf The i-th row of A is $A_{i1}e_1+\cdots+A_{in}e_n$ so we seek to compute $\det(A_{11}e_1+\cdots+A_{1n}e_n, \ldots, A_{n1}e_1+\cdots+A_{nn}e_n)$.

Using multilinearity to expand get n^n terms, each of the form $A_{1j_1}A_{2j_2}\cdots A_{nj_n}\det(e_{j_1},\ldots,e_{j_n})$ $\star$

If any $e_{j_i}=e_{j_k}$, get 0, so only permutations $\sigma=\begin{pmatrix}1&2&\cdots&n\\ j_1&j_2&\cdots&j_n\end{pmatrix}$ contribute. The matrix with rows $e_{j_1},\ldots,e_{j_n}$ is P_σ^T with $\det P_\sigma^T=\det P_\sigma=\mathrm{sgn}(\sigma)$. Thus the contribution of $\star$ is $A_{1\sigma(1)}\cdots A_{n\sigma(n)}\,\mathrm{sgn}(\sigma)$. $\square$

e.g.

$$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} \quad \begin{pmatrix}\boxed{a_{11}}&a_{12}&a_{13}\\ a_{21}&\boxed{a_{22}}&a_{23}\\ a_{31}&a_{32}&\boxed{a_{33}}\end{pmatrix} \quad a_{11}a_{22}a_{33}$$

$$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} \quad \begin{pmatrix} &\square& \\ \square& & \\ & &\square\end{pmatrix} \quad -a_{12}a_{21}a_{33}$$

$$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \quad \begin{pmatrix} & &\square\\ &\square& \\ \square& & \end{pmatrix} \quad -a_{13}a_{22}a_{31}$$

$$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \quad \begin{pmatrix}\square& & \\ & &\square\\ &\square& \end{pmatrix} \quad -a_{11}a_{23}a_{32}$$

$$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \quad \begin{pmatrix} &\square& \\ & &\square\\ \square& & \end{pmatrix} \quad a_{12}a_{23}a_{31}$$

$$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \quad \begin{pmatrix} & &\square\\ \square& & \\ &\square& \end{pmatrix} \quad +\ a_{13}a_{21}a_{32}$$

$$\overline{\phantom{a_{13}a_{21}a_{32}}}$$

det