Recall $\det : M_{n\times n}(F) \to F$ is the unique multilinear, alternating function of the rows of an $n\times n$ matrix, normalized so that $\det(I_n)=1$

Know: · swapping rows switches sign
· scaling a row scales det
· adding a scaler multiple of one row to another does nothing
· $\det A \neq 0 \iff \operatorname{rank}(A)=n \iff A$ is invertible

To do: · $\det A^T = \det A$ — Today
· $\det AB = \det A \det B$ — ~~Today~~ HW8F
· row/column expansion — Friday
· permutation expansion $\det A = \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}$ — Wed
· Over $\mathbb{R}$, $|\det A| = \operatorname{vol}(A\cdot[0,1]^n)$ — next Monday
· det exists and is unique — Friday

Elementary Matrices An $n\times n$ matrix is called an elementary matrix if it is obtained from I_n through a single row operation.

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & \lambda & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$$

Fact If E is an $n\times n$ elementary matrix and $A\in M_{n\times k}(F)$, then EA is the matrix obtained from A by performing the row operation associated with E.

Upshot You can perform row ops via mult'n by elementary matrices

e.g. $E=\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftrightsquigarrow r_2 \to r_2 - 3r_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 5 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -10 & -10 \\ 1 & 5 & 6 & 7 \end{pmatrix}$$

Note $REF(A) = E_\ell \cdots E_2 E_1 A$ for some elementary matrices E_i.

Now look at $\det A^T$ vs $\det A$.

e.g. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc.$ ✓

Thm $\det AB = \det A \cdot \det B$

Pf HW $\square$

Prop (1) $(AB)^T = B^T A^T$

(2) $(A^T)^{-1} = (A^{-1})^T$ for A invertible.

Pf (1) ✓

(2) The linear trans version says $(f^{-1})^* = (f^*)^{-1}$, which we now prove: For $f: V \to W$ linear iso with inverse $f^{-1}: W \to V$

here $f \circ f^{-1} = id_W \implies (f \circ f^{-1})^* = id_{W^*}$

$\implies (f^{-1})^* \circ f^* = id_{W^*}$

Similarly, $f^* \circ (f^{-1})^* = id_{V^*}$, so $(f^{-1})^* = (f^*)^{-1}$. $\square$

Lemma For E elementary, $\det E = \det E^T \neq 0$.

Pf (1) For E given by swapping i, j, $E = E^T$ & $\det E = -1 = \det E^T$.

(2) For E scaling of row by $\lambda \neq 0$, $E = E^T$ & $\lambda = \det E = \det E^T \neq 0$.

(3) For E given by $r_j \to r_j + \lambda r_i$, E^T given by $r_i \to r_i + \lambda r_j$

so $\det E = \det E^T = \det I_n = 1$. $\square$

Thm $\det A = \det A^T$

Pf There are elementary matrices $E_1, \dots, E_\ell$ s.t.

$REF(A) = E_\ell \cdots E_1 A$ ⊛

$\implies \det REF(A) = \det E_\ell \cdots \det E_1 \det A$

$\implies \det A = \det(E_\ell)^{-1} \cdots \det(E_1)^{-1} \det REF(A)$

Taking $\circledast^T$: $\quad REF(A)^T = A^T E_1^T \cdots E_\ell^T$.

Taking det and solving for $\det A^T$ (using $\det E_i = \det E_i^T$):

$$\det A^T = \det(E_1)^{-1} \cdots \det(E_\ell)^{-1} \det REF(A)^T$$
$$= \det(E_\ell)^{-1} \cdots \det(E_1)^{-1} \det REF(A)^T$$

Two cases: (1) $\operatorname{rank} A = n \iff REF(A) = I_n \Rightarrow \det REF(A) = 1$ and $REF(A)^T = I_n^T = I_n$ so $\det REF(A)^T = 1$ as well. Thus

$$\det A = \det(E_\ell)^{-1} \cdots \det(E_1)^{-1} = \det A^T.$$

(2) $\operatorname{rank} A < n \implies \operatorname{rank} A^T = \operatorname{rank} A < n$

$$\implies \det A = \det A^T = 0. \qquad \square$$

<u>Cor</u> det is a multilinear, alternating function of the columns of a square matrix. $\square$