So far have seen that if $\det : M_{n\times n}(F) \to F$ multilin, alternating in rows with $\det I_n = 1$ exists, then

$$\det A = \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma)\, A_{1\sigma(1)} \cdots A_{n\sigma(n)}$$

so existence will imply uniqueness.

<u>Notation</u> For $A \in M_{n\times n}(F)$ and $1 \le i,j \le n$, let $A(i|j)$ be the matrix obtained by deleting the i-th row and j-th column from A.

<u>Lemma</u> Suppose that $n > 1$ and that $D : M_{n-1\times n-1}(F) \to F$ is multilin, alt with $D(I_{n-1}) = 1$. Fix $j \in \{1,\dots,n\}$ and define $d_j : M_{n\times n}(F) \to F$ by $d_j(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij}\, D(A(i|j))$. Then d_j is alt multilin with $d_j(I_n) = 1$.

<u>Pf</u> Direct computation. $\square$

<u>Thm</u> For every $n \geq 1$ $\exists!$ det on $M_{n\times n}(F)$. Moreover, this function satisfies $\det A = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det A(i|j)$ for every $j \in \{1,\dots,n\}$ and all $A \in M_{n\times n}(F)$.

<u>Pf</u> Existence follows inductively from the lemma. Uniqueness follows from permutation expansion. Since the determinant is unique, all the fns d_j are equal. $\square$

<u>Rmk</u> This is called cofactor (or Laplace) expansion.
The (i,j) <u>cofactor</u> of A is $(-1)^{i+j} \det A(i|j) =: C_{ij}$.
We get $\det A = \sum_{i=1}^{n} A_{ij} C_{ij} = \sum_{j=1}^{n} A_{ij} C_{ij}$
(use $\det A = \det A^T$.

e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Expand along 2nd row:

$$\det A = -2\det\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} + 0\cdot\det\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= (-2)(-1) - (-1) = 3$$

Along 3rd column:

$$\det A = 3\det\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} + 1\cdot\det\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

$$= 3(2) - 1(-1) + 1(-4) = 3.$$

Rmk If a matrix has many 0's along a row or col, expand along it for quick comp'n:

$$\det\begin{pmatrix} 1 & 3 & 0 \\ 3 & 2 & 3 \\ 1 & 4 & 0 \end{pmatrix} = -3\det\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} = -3.$$

Defn For $A \in M_{n\times n}(F)$ let $C \in M_{n\times n}(F)$ have $C_{ij} = (-1)^{i+j}\det A(i|j)$ (the matrix of cofactors of A). The adjugate of A is

$$\operatorname{adj}(A) := C^T.$$

Thm $\operatorname{adj}(A)\cdot A = (\det A) I_n$.

Pf Let $B = \operatorname{adj}(A)A$. Then $B_{ii} = \sum_{k=1}^{n} \operatorname{adj}(A)_{ik} A_{ki}$

= expansion of $\det A$ along i-th col

$= \det A$.

For $i \neq j$, remains to show $B_{ij} = 0$. Let M be the matrix obtained by replacing the i-th col of A with A's j-th col. Show $\det M = B_{ij}$ so $B_{ij} = 0$. $\square$

Cor If A is invertible, $A^{-1} = (\det A)^{-1}\operatorname{adj}(A)$. $\square$