Recall $V^* = \mathcal{L}(V,F)$

$$\mathcal{L}(V,W) \longrightarrow \mathcal{L}(W^*,V^*)$$
$$\varphi \longmapsto \varphi^* = f\circ\varphi$$

Q How are $\ker\varphi$, $\operatorname{im}\varphi$ related to $\ker\varphi^*$, $\operatorname{im}\varphi^*$?

Defn Let $S\subseteq V$ be a subset of V. The annihilator of S is the subset of V^* defined by $S^\circ = \{f\in V^* \mid f(s)=0\ \forall s\in S\}$.

e.g. $V=\mathbb{R}[x]$, $S=\{p\in V \mid p(0)=0\}$. (so $S=$ multiples of x $=$ const term 0 polynomials)

For $\lambda\in\mathbb{R}$, define $f_\lambda\in V^*$ by $f_\lambda(p)=\lambda p(0)$.

Claim $S^\circ = \{f_\lambda \mid \lambda\in\mathbb{R}\}$.

Indeed, if $p\in S$ then $f_\lambda(p)=\lambda p(0)=\lambda\cdot 0=0$ so $f_\lambda\in S^\circ$.

Now suppose $g\in S^\circ$. Restricting g to $\mathbb{R}$ (viewed as const polys) gives a linear trans $g|_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}$. Let $\lambda=g(1)$. Then $g|_{\mathbb{R}}(r)=\lambda r$ $\forall r\in\mathbb{R}$. Since $g\in S^\circ$, $g(x^i)=0$ for $i>0$. Thus if $p=a_nx^n+\dots+a_0\in V$, then $g(p)=a_ng(x^n)+\dots+a_1g(x)+\cancel{a_0g(1)}\ g(a_0)$
$$=\lambda a_0 = \lambda p(0).$$

Thus $g=f_\lambda$. $\square$

Note $\{f_\lambda \mid \lambda\in\mathbb{R}\}$ is a subspace of V^*.

HW $S^\circ\subseteq V^*$ is a subspace $\forall$ subset $S\subseteq V$.

Lemma Suppose V is fin dim. Let $S\subseteq V$ be a subspace, and $i:S\to V$ be the inclusion map $i(s)=s$. Then $\operatorname{im}(i^*)=S^*$.

Pf HW $\square$

Prop For V fin dim, $S\subseteq V$ subspace,
$$\dim(S)+\dim(S^\circ)=\dim V.$$

Pf $\operatorname{rank}(i^*)+\operatorname{null}(i^*)=\dim V^*$, $\ker(i^*)=S^\circ$, $\dim V=\dim V^*$, so
$$\underbrace{\dim(S^*)}_{=\dim S}+\dim(S^\circ)=\dim V \qquad \square$$

<u>Thm</u> Suppose V, W fin dim, $\varphi \in \mathcal{L}(V,W)$. Then

(a) $\ker(\varphi^*) = \operatorname{im}(\varphi)^\circ$

(b) $\operatorname{null}(\varphi^*) = \operatorname{null}(\varphi) + \dim W - \dim V$.

<u>Pf</u> For (a), note that if $f \in W^*$, then

$$\begin{aligned} f \in \ker(\varphi^*) &\iff f \circ \varphi = 0 \\ &\iff f(\varphi(v)) = 0 \quad \forall v \in V \\ &\iff f(w) = 0 \quad \forall w \in \operatorname{im}(\varphi) \\ &\iff f \in \operatorname{im}(\varphi)^\circ . \end{aligned}$$

For (b), apply the prop to $S = \operatorname{im}(\varphi) \subseteq W$ to obtain

$$\dim \operatorname{im} \varphi + \dim \operatorname{im}(\varphi)^\circ = \dim W .$$

But $\dim \operatorname{im} \varphi = \operatorname{rank}(\varphi)$ & $\cancel{\operatorname{rank} \varphi + \operatorname{null}}$ $\dim \operatorname{im}(\varphi)^\circ = \dim \ker \varphi^* = \operatorname{null} \varphi^*$

so $$\operatorname{rank} \varphi + \operatorname{null} \varphi^* = \dim W .$$

By rank-nullity, $\operatorname{rank} \varphi = \dim V - \operatorname{null} \varphi$, so

$$\operatorname{null} \varphi^* = \operatorname{null} \varphi + \dim W - \dim V . \qquad \square$$

<u>Cor</u> φ^* is inj $\iff$ φ is surjective.

<u>Pf</u> HW $\square$

<u>Thm</u> Suppose V, W fin dim, $\varphi \in \mathcal{L}(V,W)$. Then

(a) $\operatorname{rank} \varphi^* = \operatorname{rank} \varphi$

(b) $\operatorname{im}(\varphi^*) = \ker(\varphi)^\circ$

<u>Pf</u> For (a), apply rank-nullity to φ & φ^*:

$$\begin{aligned} \operatorname{rank} \varphi^* &= \dim W^* - \operatorname{null} \varphi^* \\ \operatorname{rank} \varphi &= \dim V - \operatorname{null} \varphi \end{aligned}$$

$$\Rightarrow \operatorname{rank} \varphi^* - \operatorname{rank} \varphi = \operatorname{null} \varphi + \dim W - \dim V - \operatorname{null} \varphi^* = 0 .$$

For (b), suppose $f \in V^*$ is in the image of φ^*, so that $f = \varphi^*(g)$ for some $g \in W^*$. To show $f \in \ker(\varphi)^\circ$, must show $f(v) = 0$ $\forall v \in \ker \varphi$.

For $v \in \ker \varphi$, $f(v) = g(\varphi(v)) = g(0) = 0$ ✓ so $\operatorname{im} \varphi^* \subseteq \ker(\varphi)^\circ$.

Now check dimensions are equal, proving equality:

By the Prop, $\operatorname{null} \varphi + \dim \ker(\varphi)^\circ = \dim V$, so

$$\begin{aligned} \dim \ker(\varphi)^\circ &= \dim V - \operatorname{null} \varphi \\ &= \operatorname{rank}(\varphi) \\ &= \operatorname{rank} \varphi^* \\ &= \dim \operatorname{im} \varphi^* . \end{aligned}$$

□

<u>Cor</u> φ^* surj iff φ is inj.

<u>Pf</u> HW □