Dual Vector Spaces

Defn (1) For V an F-vector space, $V^* := \mathcal{L}(V, F)$ is the dual space of V. Elements of V^* are called linear functionals.

(2) If V is finite dimensional with basis $\{v_1, \dots, v_n\}$, define $v_i^* \in V^*$ for $i \in \{1, \dots, n\}$ by its action on $\{v_1, \dots, v_n\}$:

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

Prop $\{v_1^*, \dots, v_n^*\}$ is a basis of V^*. In particular, if $\dim V < \infty$, then $\dim V = \dim V^*$.

Pf Since $\dim V < \infty$, $\dim V^* = \dim \mathcal{L}(V, F) = \dim M_{1 \times \dim V}(F)$
$= \dim V$.

Since there are n v_i^*'s, it suffices to show they are linearly independent. If $a_1 v_1^* + \dots + a_n v_n^* = 0$, then, applying this eqn to v_j, get $a_j = 0$. This holds for all j, so the v_i^* are lin ind. $\square$

Defn $\{v_1^*, \dots, v_n^*\}$ is the dual basis of $\{v_1, \dots, v_n\}$.

Double Duals

Defn The dual of V^*, namely V^{**}, is the double dual of V.

Thm There is a natural linear injection $V \longrightarrow V^{**}$. If V is finite dimensional, then this linear transformation is an isomorphism.

Pf Let $v \in V$. Define the evaluation at v map

$$ev_v : V^* \longrightarrow F$$
$$f \longmapsto f(v).$$

Then $ev_v(f+\lambda g) = (f+\lambda g)(v) = f(v) + \lambda g(v) = ev_v(f) + \lambda ev_v(g)$

so ev_v is a linear transformation $V^* \to F$, i.e., $ev_v \in V^{**}$.

We thus get a natural map $\varphi : V \longrightarrow V^{**}$, $v \longmapsto ev_v$

and φ is linear: $ev_{v+\lambda w}(f) = f(v+\lambda w) = f(v) + \lambda f(w) = ev_v(f) + \lambda ev_w(f)$

for all $f \in V^*$, $v, w \in V$, $\lambda \in F$. Thus

$$\varphi(v+\lambda w) = ev_{v+\lambda w} = ev_v + \lambda ev_w = \varphi(v) + \lambda \varphi(w).$$

For injectivity, we punt — this requires knowing that V has a basis (containing any specified nonzero v), but we have only proven this for fin dim'l vector spaces. Nevertheless, it's true!

For fin dim'l V, given $v \neq 0 \in V$, $\exists$ basis $B \ni v$.

Define $f : V \longrightarrow F$, $v \longmapsto 1$, $B \setminus \{v\} \longmapsto 0$. Then $f \in V^*$ and $ev_v(f) = f(v) = 1$.

Thus $\varphi(v) = ev_v \neq 0 \Rightarrow \ker \varphi = \{0\}$

$\Rightarrow \varphi$ inj.

By rank-nullity, φ is an isomorphism since

$\dim V = \dim V^* = \dim V^{**}$. $\square$

<u>Dual transformations</u> and <u>transpose matrices</u>

Given $\varphi : V \to W$ linear and $f \in W^*$, we have $f \circ \varphi \in V^*$.

<u>Prop</u> The assignment $\varphi^* : W^* \longrightarrow V^*$, $f \longmapsto f \circ \varphi$ is linear.

<u>Pf</u> $\varphi^*(f+\lambda g) = (f+\lambda g)\circ\varphi = f\circ\varphi + \lambda(g\circ\varphi) = \varphi^* f + \lambda \varphi^* g$.

<u>Defn</u> $(\ )^T : M_{m\times n}(F) \longrightarrow M_{n\times m}(F)$, $(a_{ij}) \longmapsto (a_{ji}) = (a_{ij})^T$ is the <u>transpose</u>.

<u>Thm</u> Let $\alpha = \{v_1, \dots, v_n\}$, $\beta = \{w_1, \dots, w_m\}$ be ordered bases of V, W, resp.

~~Then~~ Let $A_\alpha^\beta(\varphi)$ denote the matrix of φ wrt α, β.

Let $\alpha^* = \{v_1^*, \dots, v_n^*\}$, $\beta^* = \{w_1^*, \dots, w_m^*\}$ be the dual bases.
Then $A_{\beta^*}^{\alpha^*}(\varphi^*) = A_\alpha^\beta(\varphi)^T$ for any lin trans $\varphi: V \to W$.

<u>Pf</u> An exercise in (advanced!) bookkeeping:

Let $A_\alpha^\beta(\varphi) = (a_{ij})$ so that $\varphi(v_j) = \sum_{i=1}^m a_{ij} w_i$, $1 \le j \le n$.

Now $\varphi^*(w_k^*)(v_j) = (w_k^* \circ \varphi)(v_j) = w_k^*\left(\sum_{i=1}^m a_{ij} w_i\right) = a_{kj}$.

Also $\left(\sum_{i=1}^n a_{ki} v_i^*\right)(v_j) = a_{kj}$ for all j. Thus $\varphi^*(w_k^*)$ and $\sum_{i=1}^n a_{ki} v_i^*$ agree on a basis $\Longrightarrow \varphi^*(w_k^*) = \sum_{i=1}^n a_{ki} v_i^*$.

This says that the k-th column of $A_{\beta^*}^{\alpha^*}(\varphi^*)$ is equal to the k-th row of $A_\alpha^\beta(\varphi)$ $\forall k$, so

$$A_{\beta^*}^{\alpha^*}(\varphi^*) = A_\alpha^\beta(\varphi)^T. \qquad \square$$