Determinants

Defn The determinant is a multilinear, alternating function of the rows of a square matrix, $\det: M_{n\times n}(F) \to F$, normalized so that its value on the identity matrix is 1.

To explain, for $A \in M_{n\times n}(F)$ with rows $r_1, \dots, r_n \in F^n$, write $\det(r_1, \dots, r_n)$ for $\det A$. Then

① Multilinear: The determinant is a linear fn wrt each row:

$$\det(r_1,\dots, r_{i-1}, r_i + \lambda r_i', r_{i+1}, \dots, r_n)$$
$$= \det(r_1, \dots, r_n) + \lambda \det(r_1, \dots, r_{i-1}, r_i', r_{i+1}, \dots, r_n).$$

② Alternating: The determinant is 0 if two of the rows are equal:

$$\det(r_1, \dots, r_n) = 0 \text{ if } r_i = r_j \text{ for some } i \neq j.$$

③ Normalized: $\det(I_n) = \det(e_1, \dots, e_n) = 1$.

Thm For each $n \geq 0$, $\exists!\ \det: M_{n\times n}(F) \to F$.

For now, assume det exists satisfying ①-③.

Prop [det & row ops] Let $A, B \in M_{n\times n}(F)$.

① If B is obtained from A by swapping two rows, $\det B = -\det A$.
② If B — " — by scaling a row by λ, $\det B = \lambda \det A$.
③ If B — " — by adding a scalar mult of one row to another, then $\det B = \det A$

Pf ① In the case of swapping r_1, r_2 in A to get B, compute

$$0 = \det(r_1+r_2, r_1+r_2, r_3, \dots, r_n) \quad [\text{alt}]$$
$$= \det(r_1, r_1+r_2, r_3, \dots, r_n) + \det(r_2, r_1+r_2, r_3, \dots, r_n) \quad [\text{mult}]$$
$$= \det(r_1, r_1, r_3, \dots, r_n) + \det(r_1, r_2, r_3, \dots, r_n) + \det(r_2, r_1, r_3, \dots, r_n) + \det(r_2, r_2, r_3, \dots)$$
$$= 0 + \det A + \det B + 0 \implies \det B = -\det A$$

② Implied by multilinearity.

③ $$\det(r_1, \lambda r_1 + r_2, r_3, \dots, r_n) = \lambda \cancel{\det(r_1, r_1, \dots, r_n)}^{0} + \det(r_1, r_2, \dots, r_n)$$
$$= \det(r_1, \dots, r_n)$$

e.g.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\big((a,b),\ (c,d)\big)$$

$$= \det(ae_1 + be_2,\ ce_1 + de_2)$$

$$= a\det(e_1,\ ce_1 + de_2) + b\det(e_2,\ ce_1 + de_2)$$

$$= ac\,\cancelto{0}{\det(e_1,e_1)} + ad\det(e_1,e_2) + bc\det(e_2,e_1)$$
$$+\ bd\,\cancelto{0}{\det(e_2,e_2)}$$

$$= ad\det I_2 - bc\det I_2$$

$$= ad - bc.$$

The prop turns Gauss-Jordan reduction into an algorithm for computing det!

e.g.

$$\det\begin{pmatrix} 1 & 2 & -2 \\ 9 & 4 & 0 \\ 2 & 2 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & -2 \\ 0 & -14 & 18 \\ 0 & -2 & 8 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & -2 & 8 \\ 0 & -14 & 18 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & -14 & 18 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & -38 \end{pmatrix}$$

$$= 2(-38)\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2(-38)\det I_3$$

$$= -76.$$

TPS [in groups of 4]

- What is $\det\begin{pmatrix} 4 & 2 & -3 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$?
- What is $\det\begin{pmatrix} \diagdown & \\ 0 & \end{pmatrix}$?

if time

Prop TFAE:

1. $\det A \neq 0$
2. $\operatorname{rank}(A) = n$
3. A invertible.