Matrices & Linear Transformations

$$M_{m\times n}(F) \longrightarrow \mathcal{L}(F^n, F^m)$$
$$A \longmapsto (f_A : x \longmapsto Ax) \quad \text{for } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Linearity of f_A: $f_A(u+\lambda v) = A(u+\lambda v) = Au + \lambda Av$
$$= f_A(u) + \lambda f_A(v).$$

e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\quad f_A : F^3 \to F^2$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ 4x+5y+6z \end{pmatrix}$$

Notes ① Coeffs come from corresponding row
② $f_A(e_j) = j$th column of A.

This gives us an idea on producing an inverse function
$$\mathcal{L}(F^n, F^m) \longrightarrow M_{m\times n}(F)$$
$$f \longmapsto (f(e_1) \cdots f(e_n)) \quad \text{with } f(e_j) \text{ written as a column}$$

Facts (a) $M_{m\times n}(F) \longrightarrow \mathcal{L}(F^n, F^m)$ is linear
(b) and a bij, hence an isomorphism

TPS What does it mean for $A \mapsto f_A$ to be linear?

Q How can we encode a lin trans $f: V \to W$ with a ~~lin trans~~ matrix?
(V, W fin dim)

Idea Choosing a basis for V is equiv to producing an isomorphism $V \xrightarrow{\cong} F^n$. Do this for W as well then use the above assignment.

Suppose $\alpha = \{v_1, \dots, v_n\}$ is an ordered basis of V and $v = c_1v_1 + \dots + c_nv_n$ has coords $(c_1, \dots, c_n)$. Get $\phi_\alpha : V \xrightarrow{\cong} F^n$
$$v \longmapsto (c_1, \dots, c_n).$$

Similarly, if $\beta = \{w_1, \dots, w_m\}$ basis of W, get $\phi_\beta : W \xrightarrow{\cong} F^m$.

The $m \times n$ matrix A_α^β representing f wrt these bases is the one making

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\phi_\alpha"', "\cong"] & W \arrow[d, "\cong"', "\phi_\beta"] \\ F^n \arrow[r, "f_{A_\alpha^\beta}"'] & F^m \end{tikzcd}$$

commute.

We have
$$\begin{tikzcd} v_j \arrow[r, mapsto] \arrow[d] & f(v_j) \\ e_j \arrow[r, mapsto] & j\text{th column of } A_\alpha^\beta \end{tikzcd}$$

so the j-th column of A_α^β must be the β-coords of $f(v_j)$.

I.e. $\boxed{A_\alpha^\beta = (a_{ij}) \text{ where } f(v_j) = a_{1j}w_1 + \cdots + a_{mj}w_m}$.

<u>e.g.</u> $f: \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 3}$

$p \longmapsto xp + p'$

$\alpha = \{1, x, x^2\}, \ \beta = \{1, x, x^2, x^3\}$

$f(1) = x = 0\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3$

$f(x) = x^2 + 1 = 1\cdot 1 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3$

$f(x^2) = x^3 + 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2 + 1\cdot x^3$

Thus $A_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ rep's f wrt α, β.

<u>TPS</u>
- What is dim $\mathcal{L}(V, W)$ for dim $V = n$, dim $W = m$?
- How is this related to V^*?