<u>Identity matrices</u> The $n\times n$ <u>identity matrix</u> I_n has 1's on diag, 0's elsewhere. Whenever defined, $AI = A$, $IB = B$.

<u>Inverses</u> $A \in M_{m\times n}(F)$, $B \in M_{n\times m}(F)$. If $AB = I_n$, call A a <u>left inverse</u> for B, B a <u>right inverse</u> for A.

<u>e.g.</u> $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

so A is a left-inverse for B but B is <u>not</u> a left-inverse for A.

<u>Thm</u> $A, B \in M_{n\times n}(F)$. TFAE: ① $AB = I_n$, ② $BA = I_n$.

In this case, say A, B <u>invertible</u>, $A^{-1} = B$, $B^{-1} = A$.

TFAE: ③ A is invertible, ④ $\mathrm{rank}(A) = n$, ⑤ the reduced echelon form of A is I_n.

Proof follows from an algorithm for computing inverses.

<u>Calculating the inverse</u>

An example first: Let $A = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. A right inverse to A

would satisfy $$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This reduces to 3 problems:

$$A\begin{pmatrix} a \\ d \\ g \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} b \\ e \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} c \\ f \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\left(\begin{array}{ccc|c} 0 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \end{array}\right) \quad \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \end{array}\right)$$

Combine into one "super"-augmented matrix reduction:

$$(A|I) = \left(\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 - r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{array}\right) \xrightarrow[r_3 \to -r_3]{r_2 \leftrightarrow r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 3 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_3 \to r_3 - 3r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & -4 & 1 & -3 & 3 \end{array}\right) \xrightarrow{r_3 \to -r_3/4} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{3}{4} & \frac{-3}{4} \end{array}\right)$$

$$\xrightarrow[r_2 \to r_2 - r_3]{r_1 \to r_1 - r_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1/4 & 1/4 & 3/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 3/4 & -3/4 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

This easily generalizes to $A \in M_{n \times n}(F)$.

Algorithm for computing A^{-1}:

Perform row ops to $(A | I_n)$ to compute REF of A:

$$(A | I_n) \longrightarrow (\mathrm{REF}(A) | B).$$

Then if $\mathrm{REF}(A) \neq I_n$, then $\mathrm{rank}(A) < n$ and A has no inverse. If $\mathrm{REF}(A) = I_n$, then $\mathrm{rank}(A) = n$, and B = right inverse of A.

Performing the same algorithm on $(A^T | I_n)$ computes left inverse C. (Note $\mathrm{rank}(A^T) = \mathrm{rank}(A)$.) Then $AB = I$ & $CA = I$ so

$C(AB) = CI = C \Rightarrow (CA)B = C$, but $(CA) = I$, so $IB = C$,

i.e. $B = C$, as desired. Thus the algorithm computes the 2-sided inverse $B = A^{-1}$.