Recall $M_{m\times n}(F) \xrightarrow{\cong} \mathcal{L}(F^n, F^m)$

$A \longmapsto (f_A : x \mapsto Ax)$

$A_f := (f(e_1) \cdots f(e_n)) \longleftarrow f$

Image For $f: F^n \to F^m$, $\mathrm{im}(f) = \{f(x) \mid x \in F^n\} \subseteq F^m$.

$= \mathrm{span}\{f(e_1), \dots, f(e_n)\}$

Second equality b/c $x = (x_1, \dots, x_n) = x_1e_1 + \cdots + x_ne_n$
$\longmapsto f(x) = x_1 f(e_1) + \cdots + x_n f(e_n) \in \mathrm{span}\{f(e_1), \dots, f(e_n)\}$, giving $\mathrm{im}(f) \subseteq \mathrm{span}\{f(e_1), \dots, f(e_n)\}$. The other inclusion follows b/c each $f(e_j) \in \mathrm{im}(f)$ & $\mathrm{im}(f)$ is a subspace.

Prop $\mathrm{im}(f) =$ column space of A_f

$\mathrm{rank}(f) = \mathrm{rank}(A_f)$. $\square$

e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -3 & 2 \end{pmatrix}$ then $\mathrm{im}(f_A) = \mathrm{span}\{(1,0,-3), (1,1,2)\}$

Indeed, $f_A(x,y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ -3x+2y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Composition

Thm $A \in M_{m\times p}(F)$, $B \in M_{p\times n}(F)$ with associated lin trans $f_A : F^p \to F^m$, $f_B : F^n \to F^p$. Then

$$f_A \circ f_B = f_{AB}.$$

Pf For $x \in F^n$, $(f_A \circ f_B)(x) = f_A(f_B(x)) = A(Bx) = (AB)x$

$= f_{AB}(x)$. $\square$

Note Matrix multn was invented so that this would happen.

Inverses Suppose $A, B \in M_{n \times n}(F)$, $AB = I_n$.

TPS What is f_{I_n}?

Get $f_A \circ f_B = id_{F^n}$. ~~It follows that f_B is injective so has kernel $\{0\}$ and thus $\text{rank}(f_B) = \text{rank}$~~

It follows that f_A is surjective $\Rightarrow \text{rank}(f_A) = \text{rank}(A) = n$.
In particular, $\text{rank}(A) < n \Longrightarrow A$ doesn't have a right inverse.
Similarly, f_B injective $\Rightarrow \text{nullity}(f_B) = 0 \Longrightarrow \text{rank}(f_B) = \text{rank}(B) = n$
so $\text{rank}(B) < n \Longrightarrow B$ doesn't have a left inverse.
This completes our argument about matrix inversion! $\square$

Time allowing How do kernels fit into this picture?