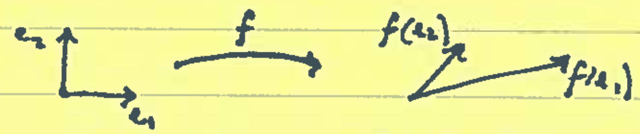
Geometry of Linear Transformations

Goal: Build visual intuition for linear trans $f: \mathbb{R}^m \to \mathbb{R}^n$, focusing on $m=n=2$.

Recall that $f: V \to W$ linear is specified by its action on a basis of V: Suppose $b_1, \dots, b_n$ form a basis of V. For any $w_1, \dots, w_n \in W$, $\exists!$ lin trans $f: V \to W$ s.t. $f(b_i) = w_i$. (Then $f(\Sigma a_i b_i) = \Sigma a_i f(b_i) = \Sigma a_i w_i$.)

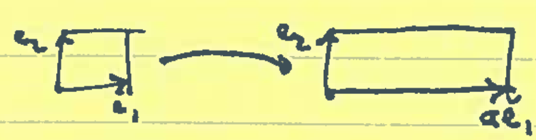
In particular a linear trans $f: \mathbb{R}^2 \to \mathbb{R}^2$ is specified by $f(e_1) = f(1,0)$ & $f(e_2) = f(0,1)$.



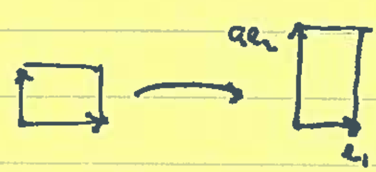
$$(a,b) \longmapsto af(e_1) + bf(e_2)$$

Thus it is common to visualize linear trans $f: \mathbb{R}^2 \to \mathbb{R}^2$ by what they do to the unit square $[0,1] \times [0,1] \subseteq \mathbb{R}^2$. Here are some special cases along with their effects on $[0,1]^2$:

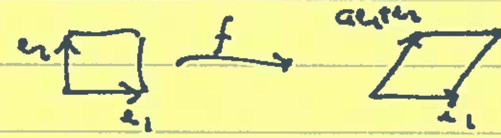
Scale
- $e_1 \mapsto ae_1$, $e_2 \mapsto e_2$



- $e_1 \mapsto e_1$, $e_2 \mapsto ae_2$

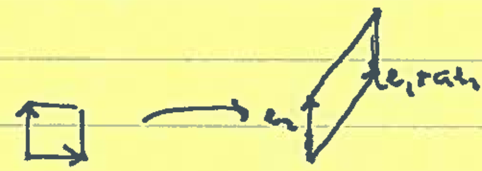


Shear
- $e_1 \mapsto e_1$
- $e_2 \mapsto ae_1 + e_2$

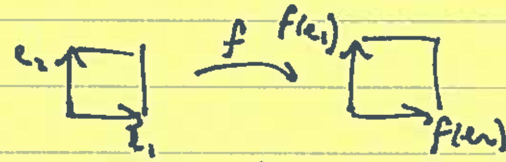


- $e_1 \mapsto e_1 + ae_2$

  $e_2 \mapsto e_2$



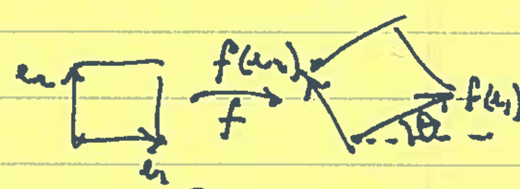
Reflect
- $e_1 \mapsto e_2$
- $e_2 \mapsto e_1$



Reflects through the $y=x$ line.

Rotate
- $e_1 \mapsto (\cos\theta, \sin\theta)$

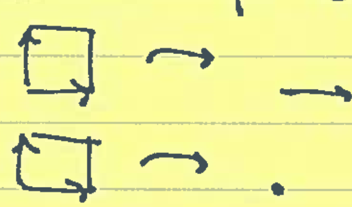
  $e_2 \mapsto (-\sin\theta, \cos\theta)$



Call this map R_θ

Squash
- $e_1 \mapsto e_1$

  $e_2 \mapsto 0$
- 0-map



Fact: Every linear trans $\mathbb{R}^m \to \mathbb{R}^n$ is the composition of a possible squash followed by shears, scales and reflections.

Note: We will prove this when studying matrix inversion.

TPS: Why is $f \circ g$ linear when f, g are linear?

Q: How can we represent R_θ as such a composition?

Special case: $\theta = \pi$. Then $e_1 \mapsto -e_1$, $e_2 \mapsto -e_2$ is clearly the composition of two scales.

Now suppose $\theta \notin k\pi$, $k \in \mathbb{Z}$.

<u>Claim</u> $R_\theta = X_\alpha \circ Y_\beta \circ X_\alpha$ for X_α an x-shear by α $(e_1 \mapsto e_1, e_2 \mapsto \alpha e_1 + e_2)$

and Y_β a y-shear by β $(e_1 \mapsto e_1 + \beta e_2, e_2 \mapsto e_2)$

with $\alpha = \gamma = -\tan(\theta/2)$, $\beta = \sin\theta$.

Indeed,

$$\begin{aligned} X_\alpha Y_\beta X_\alpha(e_1) &= X_\alpha Y_\beta(e_1) \\ &= X_\alpha(e_1 + \beta e_2) \\ &= e_1 + \beta\alpha e_1 + \beta e_2 \\ &= (1+\beta\alpha)e_1 + \beta e_2 \end{aligned}$$

$$\begin{aligned} X_\alpha Y_\beta X_\alpha(e_2) &= X_\alpha Y_\beta(\alpha e_1 + e_2) \\ &= X_\alpha(\alpha e_1 + \alpha\beta e_2 + e_2) \\ &= X_\alpha(\alpha e_1 + (1+\alpha\beta)e_2) \\ &= \alpha e_1 + (1+\alpha\beta)(\alpha e_1 + e_2) \\ &= (2\alpha + \alpha^2\beta)e_1 + (1+\alpha\beta)e_2 \end{aligned}$$

Now for $\cancel{\alpha = -\tan(\theta/2)}$, $\beta = \sin\theta$, have

$$\begin{aligned} 1+\alpha\beta = \cos\theta &\iff 1 + \alpha\sin\theta = \cos\theta \\ &\iff \alpha = \frac{\cos\theta - 1}{\sin\theta} \\ &\iff \alpha = -\tan(\theta/2) \quad \text{(by trigonometry)} \end{aligned}$$

Finally, $2\alpha + \alpha^2\beta = \alpha(1 + (1+\alpha\beta)) = \alpha(1+\cos\theta) = \cancel{\ } \cos\theta$ (more trig).

TPS Express reflection through $y = -x$ as a comp'n of scale, shear, & reflect transformations.