<u>Recall</u> $f: V \to W$ linear

$\ker(f) = \{v \in V \mid f(v) = 0\}$

$\operatorname{im}(f) = \{f(v) \mid v \in V\}$

$\operatorname{rank}(f) = \dim \operatorname{im}(f)$

$\operatorname{nullity}(f) = \dim \ker(f)$

$\operatorname{rank}(f) + \operatorname{nullity}(f) = \dim V$ (Rank-nullity thm)

<u>Prop</u> Linear $f: V \to W$ is injective iff $\ker(f) = \{0\}$.

<u>Pf</u> ~~If Ex~~ By linearity, $f(0) = 0$, so if f is injective, then $\ker(f) = \{0\}$. Now suppose $\ker(f) = \{0\}$ and that $f(u) = f(v)$. Then

$$0 = f(u) - f(v) = f(u-v) \Rightarrow u - v \in \ker(f) = \{0\}$$
$$\Rightarrow u - v = 0$$
$$\Rightarrow u = v$$

so f is inj. $\square$

<u>Prop</u> Let $S \subseteq V$, $f: V \to W$ linear.

① If S is lin dep, then $f(S) = \{f(s) \mid s \in S\} \subseteq W$ is lin dep.

② If f is injective and S is lin ind, then $f(S) \subseteq W$ is lin ind.

<u>Pf</u> Suppose $\sum a_i s_i = 0$ for some $a_i \in F$, $s_i \in S$. Since f is linear,

$0 = f(0) = f(\sum a_i s_i) = \sum a_i f(s_i)$ so f preserves dependencies.

Now suppose f inj, S lin ind. If $0 = \sum a_i f(s_i)$ for some $a_i \in F$, $f(s_i) \in f(S)$, then, by linearity of f, $0 = f(\sum a_i f(s_i))$, i.e., $\sum a_i s_i \in \ker(f) = \{0\}$. Thus $\sum a_i s_i = 0$ and since S is lin ind, each $a_i = 0$. Thus $f(S)$ lin ind. $\square$

<u>Defn</u> A linear trans $f: V \to W$ is an <u>isomorphism</u> if $\exists$ lin trans $g: W \to V$ s.t. $g \circ f = id_V$ and $f \circ g = id_W$. (Call g the <u>inverse</u> of f, write $g = f^{-1}$.) Write $f: V \cong W$ or $V \cong W$.

Note When a function $f: A \to B$ has an inverse, we know it is bijection, so every linear isomorphism is a bijection. In fact, if $f: V \to W$ is a linear bijection, then its inverse function is also ~~bijection~~ linear (check!), and so f is an isomorphism.

Takeaway: Isomorphism = linear bij'n.

e.g. $M_{2\times 2}(F) \longrightarrow F^4$ is an isomorphism.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a,b,c,d)$$

So is $M_{m\times n}(F) \longrightarrow F^{mn}$ discussed previously.

Prop Linear $f: V \to W$ is an iso iff $\ker(f) = \{0\}$ and $\mathrm{im}(f) = W$.

Pf $\ker(f) = \{0\}$ iff f is inj, $\mathrm{im}(f) = W$ iff f is surj, so both conditions (taken together) are equivalent to f bij, i.e. f an iso. $\square$

Thm Let $\dim V = n < \infty$. Then $V \cong F^n$.

Pf Choose a basis $b_1, \dots, b_n$ of V and let $e_1, \dots, e_n$ be the standard basis of F^n. Define $f: V \to F^n$ by $f(b_i) = e_i$, $1 \le i \le n$, and extending linearly. Then $f(v) = f(\sum a_i b_i) = \sum a_i e_i = (a_1, \dots, a_n) \in F^n$ — this is the map taking v to its coordinates wrt $b_1, \dots, b_n$ which is clearly a bij'n. $\square$

Cor Let V, W be finite dimensional vector spaces. Then V and W are isomorphic iff $\dim V = \dim W$.

Pf First suppose $f: V \cong W$ and let $b_1, \dots, b_n$ be a basis of V. Then $f(b_1), \dots, f(b_n)$ are lin ind (by Prop) and they span W b/c f is surj. Thus $f(b_1), \dots, f(b_n)$ is a basis of W $\Rightarrow \dim W = n = \dim V$.

Now suppose $\dim V = \dim W = n$. By the Thm, there are isos $V \xrightarrow[\cong]{f} F^n \xleftarrow[\cong]{g} W$. Then $g^{-1} \circ f: V \to W$ is an iso. $\square$

Note For $n = 0, 1, 2, \dots$ get only one isomorphism class of n-dim vector space. Choosing an iso $V \to F^n$ is equivalent to choosing a basis of V.