Matrices

Recall $M_{m\times n}(F)$ = $m\times n$ matrices A w/ entries $A_{ij}\in F$ (entry in i-th row, j-th column)

e.g. $A=\begin{pmatrix}1&2&6\\7&0&-1\end{pmatrix}\in M_{2\times3}(\mathbb{Q})$
has $A_{1,3}=6$.

$M_{m\times n}(F)$ is an F-vs with entry-wise add'n & scalar mult;
its dimension is mn w/ basis $\{E(i,j) \mid 1\le i\le m,\ 1\le j\le n\}$

$E(i,j)$: 1 in ij posn, 0 o/w

Now define a product on matrices

$$M_{m\times p}(F)\times M_{p\times n}(F)\longrightarrow M_{m\times n}(F)$$

\# cols on left = \# rows on right

$$(AB)_{ij}=\sum_{k=1}^{p}A_{ik}\cdot B_{kj}$$

A_{ik}: steps through i-th row of A

B_{kj}: steps through j-th col of B

e.g. $\begin{pmatrix}\boxed{1\ \ 0\ \ 2}\\3\ \ 1\ \ 4\end{pmatrix}\begin{pmatrix}1&\boxed{0}\\3&\boxed{-1}\\5&\boxed{3}\end{pmatrix}=\begin{pmatrix}11&\boxed{6}\\26&11\end{pmatrix}$ $\qquad (AB)_{12}=\sum_{k=1}^{3}A_{1k}B_{k2}$

$$=1\cdot0+0\cdot(-1)+2\cdot3=6$$

This is the "dot product" of i-th row w/ j-th column
where $(a_1,\dots,a_p)\cdot(b_1,\dots,b_p)=\sum_{k=1}^{p}a_kb_k$.

Prop Let $A\in M_{m\times p}(F)$, $B,B'\in M_{p\times n}(F)$, $C\in M_{n\times q}(F)$, $\lambda\in F$, $D,D'\in M_{r\times m}(F)$. Then

(a) $\lambda(AB)=(\lambda A)B=A(\lambda B)$

(b) $A(BC)=(AB)C$

(c) $A(B+B')=(AB)+(AB')$

(d) $(D+D')A=DA+D'A$

Pf 1 of (b)

$$
\begin{aligned}
(A(BC))_{ij} &= \sum_{k=1}^{p} A_{ik} (BC)_{kj} \\
&= \sum_{k=1}^{p} A_{ik}\left(\sum_{l=1}^{n} B_{kl} C_{lj} \right) \\
&= \sum_{k=1}^{p} \sum_{l=1}^{n} A_{ik}(B_{kl} C_{lj}) \\
&= \sum_{l=1}^{n} \sum_{k=1}^{p} A_{ik}(B_{kl} C_{lj}) \\
&= \sum_{l=1}^{n} \sum_{k=1}^{p} (A_{ik} B_{kl}) C_{lj} \\
&= \sum_{l=1}^{n} \left(\sum_{k=1}^{p} A_{ik} B_{kl} \right) C_{lj} \\
&= \sum_{l=1}^{n} (AB)_{il} C_{lj} \\
&= ((AB)C)_{ij} \qquad \square
\end{aligned}
$$

Pf 2 of (b) We will build a dictionary (bij'n) — in fact, linear iso

$$
\begin{aligned}
M_{m \times n}(F) &\longleftrightarrow \mathcal{L}(F^n, F^m) \\
A &\longleftrightarrow (x \longmapsto Ax) \quad \text{(for } x \text{ col vector of length } n\text{)} \\
\text{mult} &\longleftrightarrow \text{composition}
\end{aligned}
$$

Composition is associative, so matrix mult is as well! $\square$

⚠ Matrix mult'n is not commutative! (Even when defined)

TPS How does matrix rank interact w/ scalar mult, add'n?