Linear Transformations

Q How are vector spaces related? A By linear transformations

Defn V, W F-vector spaces. A linear transformation from V to W is a function $f: V \to W$ s.t. $\forall v, v' \in V, \lambda \in F$,

$$f(v+v') = f(v) + f(v') \quad \& \quad f(\lambda v) = \lambda f(v).$$

"f preserves addition" "f preserves scalar multn"

"f preserves linear structure"

Note $f: V \to W$ is a lin trans iff $f(v + \lambda v') = f(v) + \lambda f(v')$ $\forall v, v', \lambda$.

Synonyms linear map, homomorphism

e.g. $f: \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (2x+3y, x+y-3z)$ is linear:

$$\begin{aligned} f((x,y,z) + (x',y',z')) &= f(x+x', y+y', z+z') \\ &= (2(x+x') + 3(y+y'), x+x'+y+y'-3(z+z')) \\ &= (2x+3y, x+y-3z) + (2x'+3y', x'+y'-3z') \\ &= f(x,y,z) + f(x',y',z'), \end{aligned}$$

$$f(\lambda(x,y,z)) = f(\lambda x, \lambda y, \lambda z) = \cdots = \lambda f(x,y,z).$$

TPS Is $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ linear?

Prop If $f: V \to W$ is linear, then $f(0) = 0$ (i.e. $f(0_V) = 0_W$).

Pf Since f is linear, $f(0_V) = f(0 \cdot 0_V) = 0 \cdot f(0_V) = 0_W$. □

Prop Let V, W be F-vector spaces, $B \subseteq V$ a basis. For each $b \in B$, take $w_b \in W$. Then $\exists!$ linear trans $f: V \to W$ s.t. $f(b) = w_b$ $\forall b \in B$.

Slogan Linear transformations are determined by their action on bases.

Pf Given $v \in V$, have a unique expression $v = a_1 b_1 + \cdots + a_k b_k$ for some $a_i \in F^\times$, $b_i \in B$. Define $f(v) = a_1 w_{b_1} + \cdots + a_k w_{b_k}$. Well-definition follows from uniqueness of the expression for v. Linearity forces this defn since $f(b) = w_b$. $\square$

Terminology Say f defined on B and extended linearly to V.

(For V, W F-vss, let $\mathcal{L}(V,W) = \mathrm{Hom}(V,W) = \mathrm{Hom}_F(V,W)$ be the set of linear transformations $V \to W$. This forms a vector space via the operations $(f+g)(v) = f(v) + g(v)$, $(\lambda f)(v) = \lambda(f(v))$.)

e.g. $h : \mathbb{R}^2 \to \mathbb{R}^2$ linear with $h(e_1) = (-1,1)$, $h(e_2) = (3,4)$.

Then $h(a,b) = h(ae_1 + be_2) = ah(e_1) + bh(e_2) = (-a,a) + (3b,4b)$

$= (3b-a, 4b+a)$

e.g. $\pi_i : F^n \longrightarrow F$

$$e_j \longmapsto \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

TPS What is a formula for $\pi_i(a_1, \ldots, a_n)$?

TPS Is matrix transpose $M_{2\times 2}(F) \longrightarrow M_{2\times 2}(F)$ linear?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$