Rank of matrices

Defn Let A be an $m\times n$ matrix over F. The row space of A is the subspace of F^n spanned by the rows of A. The column space of A is the subspace of F^m spanned by the columns of A.
The row rank of A is the dimension of its row space. The column rank of A is the dimension of its column space.

Note Row operations = linear combos of rows. So if $A \to B$ via row ops, then Rowspace$(B) \subseteq$ Rowspace(A). But we can reverse row ops to get $B \to A$ so the opposite inclusion holds as well. Thus:

Lemma If A, B related by row ops, then they have the same row space. In particular, the reduced echelon form of A has the same row space as A. $\square$

TPS Why are the nonzero rows of a reduced echelon matrix lin ind?

Prop Let A be an $m\times n$ matrix which reduces to E in reduced echelon form. Then the nonzero rows of E for a basis of the row space of A. $\square$

e.g. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2/3 & -4 \\ 0 & 1 & -1/3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in reduced echelon form

so $\{(1,0,2/3,-4), (0,1,-1/3,4)\}$ is a basis of the row space of A.

Lemma Row ops don't change the column rank of A.

Pf Suppose A is an $m\times n$ matrix with relation

$$c_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + c_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0$$

among its columns. This rel'n is equiv to a solution $(c_1, \dots, c_n)$ to the linear system

$$\begin{aligned} c_1 a_{11} + \cdots + c_n a_{1n} &= 0 \\ &\vdots \\ c_1 a_{m1} + \cdots + c_n a_{mn} &= 0 \end{aligned}$$

Row ops don't change sol'ns, so don't change rel'ns among cols. $\square$

We see that rel'ns among columns correspond to rel'ns b/w cols of reduced echelon form of the matrix. The cols containing a pivot form a basis, so the corr cols in A form a basis of its column space!

⚠ Must take the corresponding cols in A, not the cols in E.

e.g. In the previous example, the first 2 cols were pivot cols of E, so $\begin{pmatrix}1\\3\\7\end{pmatrix}, \begin{pmatrix}2\\3\\8\end{pmatrix}$ form a basis of col space of A.

Thm The row rank of a matrix is equal to its column rank.

Pf Let E be the reduced echelon form of a matrix A.
The number of nonzero rows equals the number of pivot columns. $\square$

Defn The rank of a matrix A, denoted rank(A), is the dimension of its row or column space.

Thm Suppose we have a homogeneous system of linear eq'ns

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Let A be the corresponding matrix. Then the vector space of solutions has dimension $n - \text{rank}(A)$. (So unique sol'n iff $\text{rank}(A) = n$.)

Pf To solve, we compute REF of A. The number of free variables $=$ # non-pivot columns $= n - \text{rank}(A)$. $\square$

TPS What about a non-homogeneous system?