Kernel and Image

Lemma Let $f: V \to W$ be a linear trans. For any subspace $U \subseteq V$, $f(U) = \{f(u) \mid u \in U\}$ is a subspace of W.

Pf Since $0 \in U$ and $f(0) = 0$, $0 \in f(U)$. Now for $f(u), f(u') \in f(U)$, $f(u) + \lambda f(u') = f(u + \lambda u') \in f(U)$ since U is closed under linear combos. $\square$

Defn The image (or range space) of $f: V \to W$ is
$$\operatorname{im}(f) = \mathcal{R}(f) = f(V) = \{f(v) \mid v \in V\}.$$
The dimn of $\operatorname{im}(f)$ is the rank of f.

e.g. $\frac{d}{dx}: F[x]_{\leq 3} \to F[x]$ has image $F[x]_{\leq 2}$

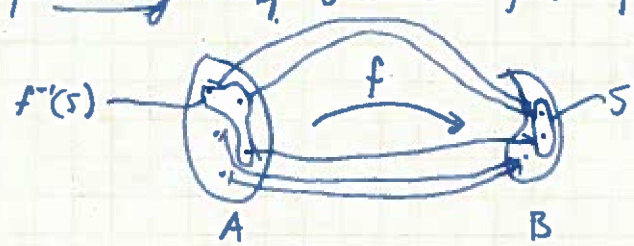
$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \longmapsto a_1 + 2a_2 x + 3a_3 x^2$$

and rank 2.

e.g. $h: M_{2\times 2}(F) \longrightarrow F[x]_{\leq 3}$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a+b+2d) + cx^2 + cx^3$$

Any vector in $\operatorname{im}(h)$ has any const term, 0 linear coeff, and equal x^2, x^3 coeff. So $\operatorname{im}(h) = \{r + sx^2 + sx^3 \mid r, s \in F\}$ and $\operatorname{rank}(h) = 2$.

Recall If $f: A \to B$ is a function and $S \subseteq B$, then the preimage of S under f is $f^{-1}(S) := \{a \in A \mid f(a) \in S\}$.



Lemma For any $f: V \to W$ linear and $U \subseteq W$ subspace, $f^{-1}(U)$ is a subspace of V.

Pf $0 \in f^{-1}(U)$ b/c $0 \in U$ and $f(0)=0$. If $v, v' \in f^{-1}(U)$, then $f(v+v') = f(v)+f(v') \in U$ b/c $f(v), f(v') \in U$. Moreover, $f(\lambda v) = \lambda f(v) \in U$ b/c $f(v) \in U$. Thus $v+v', \lambda v \in f^{-1}(U)$ so $f^{-1}(U)$ is a subspace.

Defn The kernel (or null space) of a linear map $f: V \to W$ is the inverse image of $\{0\}$,

$$\ker(f) = \mathcal{N}(f) = f^{-1}(\{0\}) = \{v \in V \mid f(v) = 0\}.$$

Note $\{0\} \subseteq W$ is a subspace so $\ker(f)$ is a subspace of V. The dimension of $\ker(f)$ is the map's nullity.

e.g. $\ker\left(\frac{d}{dx}: F[x] \to F[x]\right) = \{\text{constant polynomials}\}$ so $\frac{d}{dx}$ has nullity 1.

e.g. For $h: M_{2\times 2}(F) \to F[x]_{\leq 3}$ as before,

$$\ker(h) = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{-(a+b)}{2} \end{pmatrix} \middle| a, b \in F \right\}$$ so h has nullity 2.

e.g. $\pi_i: \mathbb{R}^n \to \mathbb{R}$ projection onto ith coord has

$$\ker(\pi_i) = \{(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) \mid a_j \in \mathbb{R}\}$$

has nullity $n-1$.

Thm If $f: V \to W$ linear, then $\operatorname{rank}(f) + \operatorname{nullity}(f) = \dim V$.

e.g. Check for previous examples.

Pf Let $\{v_1, \dots, v_k\}$ be a basis for $\ker(f)$. Extend to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V. We show ~~B~~ $\{f(v_{k+1}), \dots, f(v_n)\}$ is a basis of $\operatorname{im}(f)$, and the thm then follows.

Suppose $0 = c_{k+1} f(v_{k+1}) + \dots + c_n f(v_n)$. Then $0 = f(c_{k+1} v_{k+1} + \dots + c_n v_n)$ so $c_{k+1} v_{k+1} + \dots + c_n v_n \in \ker(f)$. Since $\{v_1, \dots, v_k\}$ is a basis

of $\ker(f)$ and $\{v_1, \dots, v_n\}$ is a basis of V, get $c_{k+1} = \cdots = c_n = 0$.
Thus B is lin ind.

Now suppose $f(v) \in \operatorname{im}(f)$. Have $v = c_1 v_1 + \cdots + c_n v_n$ and
$f(v) = c_1 f(v_1) + \cdots + c_n f(v_n) = c_{k+1} f(v_{k+1}) + \cdots + c_n f(v_n)$ since
$v_1, \dots, v_k \in \ker(f)$. Thus B spans $\operatorname{im}(f)$. $\square$