Dimension

Defn V is *finite dimensional* if it has a basis with a finite number of elements.

e.g. $\cdot\ F^n$, $M_{m\times n}(F)$ are fin dim'l

$\cdot\ F[x]$, $\mathbb{R}^{\mathbb{R}}$ are infinite dim'l.

Thm If V is finite dimensional, then every basis of V contains the same number of elts.

We'll get to the proof....

Defn If V is fin dim'l, the *dimension* of V, denoted $\dim V$ or $\dim_F V$, is the number of elements in any of its bases.

Exchange Lemma Suppose $\mathcal{B}=\{v_1,\dots,v_n\}$ is a basis for V, and suppose $w=a_1v_1+\cdots+a_nv_n\in V$ with $a_i\in F$, $a_\ell\neq 0$. Let $\mathcal{B}'=(\mathcal{B}\setminus\{v_\ell\})\cup\{w\}$. Then $\mathcal{B}'$ is also a basis of V.

Pf First show $\mathcal{B}'$ is lin ind. WLOG, $\ell=1$. Suppose $bw+b_2v_2+\cdots+b_nv_n=0$. Substituting for w,

$$0=b(a_1v_1+\cdots+a_nv_n)+b_2v_2+\cdots+b_nv_n$$
$$=ba_1v_1+(ba_2+b_2)v_2+\cdots+(ba_n+b_n)v_n.$$

Since the v_i are lin ind, $ba_1=ba_2+b_2=\cdots=ba_n+b_n=0$.

Since $a_1\neq 0$, get $b=0$, so $b_2=\cdots=b_n=0$ as well. Thus $\mathcal{B}'$ lin ind.

Now show $\mathcal{B}'$ spans V. First note $v_1=\frac{1}{a_1}w-\frac{a_2}{a_1}v_2-\cdots-\frac{a_n}{a_1}v_n$.

Take $v\in V$. Since $\mathcal{B}$ spans, $v=c_1v_1+\cdots+c_nv_n$

$$=c_1\left(\frac{1}{a_1}w-\frac{a_2}{a_1}v_2-\cdots-\frac{a_n}{a_1}v_n\right)+c_2v_2+\cdots+c_nv_n$$

$$=\frac{c_1}{a_1}w+\left(c_2-\frac{c_1a_2}{a_1}\right)v_2+\cdots+\left(c_n-\frac{c_1a_n}{a_1}\right)v_n$$

so $\mathcal{B}'$ spans V. $\square$

Thm In a finite-dimensional vector space, every basis has the same number of elements.

Pf Let V be a fin dim vector space. Among bases for V, let $B = \{u_1, \dots, u_n\}$ be one of minimal size. Let $C = \{w_1, w_2, \dots\}$ be any other basis. Know $|B| \leq |C|$, and want to show $|B| = |C|$.

Let $B_0 = B$ and consider $w_1 \in C$. By the exchange lemma, get a new basis B_1 by swapping w_1 with some u_ℓ. Relabeling if necessary, may assume $\ell = 1$ so $B_1 = \{w_1, u_2, \dots, u_n\}$.

Now consider $w_2 \in C$. Have $w_2 = a_1 w_1 + a_2 u_2 + \cdots + a_n u_n$ since B_1 is a basis. Since w_1, w_2 are lin ind, at least one of $a_2, \dots, a_n$ is nonzero. (Make sure you understand this step!) WLOG, $a_2 \neq 0$, so by exchange lemma, $B_2 = \{w_1, w_2, u_3, \dots, u_n\}$ is a basis. Continuing in this way, eventually get $B_n = \{w_1, \dots, w_n\}$ basis, $\subseteq C$.

In fact, $B_n = C$: if $w_{n+1} \in C \setminus B_n$, then $w_{n+1} = \sum_{i=1}^{n} d_i w_i$, (b/c B_n basis) but that can't happen b/c C is a basis. Thus $C = B_n$ has n elements. $\square$

Cor Let V be a fin dim vs, $S \subseteq V$ lin ind. Then S can be completed to form a basis of V.

Pf If $V \neq \operatorname{span}(S)$, then for any $v \in V \setminus \operatorname{span}(S)$, $S \cup \{v\}$ is lin ind. Continue until the set spans V. This terminates since o/w we would get an infinite basis. $\square$

Cor V fin dim vs, $V = \operatorname{span}(S)$. Then $\exists T \subseteq S$ which is a basis.

Pf Similar. $\square$

Cor A collection of n vectors in an n-dim'l vector space is lin ind $\iff$ it spans V.

Pf ($\Rightarrow$) Suppose $S \subseteq V$ lin ind, $|S| = n$. We can complete S to a basis B, but if that involves adding any vectors to it, then $|B| > n$ ☡.

($\Leftarrow$) If S spans V, $|S| = n$, then we can shrink S to a basis B, but if that involves removing any vectors, then $|B| < n$ ☡. $\square$

<u>Moral</u> Basis = min'l spanning set
= max'l lin ind set

<u>e.g.</u> (1) $\mathbb{R}^n$ has basis $\{e_1, \dots, e_n\}$

(2) $\{(1,0,0), (1,2,0), (1,2,3)\} \subseteq \mathbb{R}^3$ lin ind $\Rightarrow$ basis.

(3) $\mathbb{R}[x]_{\leq 2}$ = const, lin, or quad real polys. Basis $\{1, x, x^2\}$.

Sim, $\{1, 1+2x, 1+2x+3x^2\}$ is a basis