V an F-v.s. Recall $u_1, \dots, u_n \in V$ linearly independent when
$$a_1u_1 + \cdots + a_nu_n = 0,\ a_i \in F \implies a_1, \dots, a_n = 0.$$
A subset $S \subseteq V$ is lin ind if each of its finite subsets is lin ind.

Thm $S \subseteq V$ lin ind, $v \in \operatorname{span}(S)$. Then v can be expressed uniquely as a linear combo of elts of S.

Prop If $S_1 \subseteq S_2 \subseteq V$ and S_2 is lin ind, then S_1 is lin ind.

Pf Suppose $\sum_{i=1}^{n} a_iu_i = 0$ for some $u_i \in S_1$, $a_i \in F$. Since $S_1 \subseteq S_2$, $u_i \in S_2$ as well. Lin ind of $S_2 \implies a_i = 0\ \forall i$. $\square$

e.g. $V = (\mathbb{Z}/3\mathbb{Z})^3$, a $\mathbb{Z}/3\mathbb{Z}$-vector space.

Note that $|V| = 3^3 = 27$.

Check that $W = \{(x_1,x_2,x_3) \in V \mid x_1+x_2+x_3 = 0\} \subseteq V$ is a subspace.

Then $W = \{(-x_2-x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{Z}/3\mathbb{Z}\}$ has 9 elements.

Find a lin ind generating set:

Take $v_1 = (2,1,0) \in W$ with $\operatorname{span}\{v_1\} = \{(0,0,0), (2,1,0), (1,2,0)\}$.

Then $v_2 = (1,1,1) \in W \setminus \operatorname{span}\{v_1\}$ so $\{v_1, v_2\}$ is lin ind.

Every element of $\operatorname{span}\{v_1,v_2\}$ has a unique expression of the form $a_1v_1 + a_2v_2$ with $a_1, a_2 \in \mathbb{Z}/3\mathbb{Z}$. Thus $|\operatorname{span}\{v_1,v_2\}| = 9$.

Also $\operatorname{span}\{v_1,v_2\} \subseteq W$, and cardinalities match, so they are equal.

Bases

Defn A subset $B \subseteq V$ is a basis if it is lin ind and spans V.

An ordered basis is a basis whose elements have been listed as a sequence, $B = \{b_1, b_2, \dots\}$.

(!) The book does not distinguish b/w unordered and ordered bases (its bases are always ordered) but we will!

Prop If B is a basis of V, then every element of V can be expressed uniquely as a linear combo of elements of B.

Pf We have already seen that for B lin ind, every elt of span B has a unique such expression. Since span $B = V$, we are done. $\square$

Defn Let $B = \{v_1, \dots, v_n\}$ be an ordered basis of V. Given $v \in V$, there are unique $a_1, \dots, a_n \in F$ s.t. $v = a_1 v_1 + \dots + a_n v_n$.
The coordinates of v with respect to B are the components of the vector $(a_1, \dots, a_n) \in F^n$.

e.g. ① $B = \{e_1, e_2, e_3\} \subseteq F^3$, where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.
The coordinates of (x,y,z) wrt B are x, y, z.

② Let $e_1' = e_3$, $e_2' = e_2$, $e_3' = e_1$ and $B' = \{e_1', e_2', e_3'\}$. Then the coords of (x,y,z) wrt B' are z, y, x.

③ $B'' = \{(1,0,0), (1,1,0), (1,1,1)\}$ is an ordered basis of F^3.
Since $(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$,
(x,y,z) has coords $x-y,\ y-z,\ z$ wrt B''.

④ $V = M_{2\times 2}(F)$. Then $B = \{M_1, M_2, M_3, M_4\}$ with

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is an ordered basis of V wrt which the coords of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are a, b, c, d.

⑤ $(7,-6) = 2\cdot(5,3) - 3(1,4)$: