Condorcet's Paradox

Candidates A, B, C; 29 voters

$A>B>C : 5$
$A>C>B : 4$
$B>A>C : 2$
$B>C>A : 8$
$C>A>B : 8$
$C>B>A : 2$

$\Longrightarrow$

$A>B: +5$ $(17-12)$
$B>C: +1$ $(15-14)$
$C>A: +7$ $(18-11)$

i.e.
$$\begin{tikzcd} & A \arrow[dl, "7"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "5"'] \end{tikzcd}$$
a voting paradox or Condorcet cycle.

With head-to-head voting, any outcome can be achieved — the vote scheduler is a dictator!

<u>Goal</u> Use linear algebra to understand how/when such cycles arise.

$$V = \left\{ \begin{tikzcd} & A \arrow[dl, "c"'] & \\ C \arrow[rr, "b"'] & & B \arrow[ul, "a"'] \end{tikzcd} \;\middle|\; a,b,c \in \mathbb{R} \right\} \text{ "=" } \mathbb{R}^3$$

An $A>B>C$ voter corresponds to
$$\begin{tikzcd} & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "1"'] \end{tikzcd}$$
, etc.

The above example amounts to
$$5 \cdot \begin{tikzcd} & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "1"'] \end{tikzcd} + 4 \cdot \begin{tikzcd} & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "1"'] \end{tikzcd} + \cdots + 2 \cdot \begin{tikzcd} & A \arrow[dl, "1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "-1"'] \end{tikzcd}$$

Call a vector in $\mathbb{R}^3$ <u>purely cyclic</u> if it is of the form (k,k,k), $k \in \mathbb{R}$

Let $C = \{(k,k,k) \mid k \in \mathbb{R}\} = \left\{ \begin{tikzcd} & A \arrow[dl, "k"'] & \\ C \arrow[rr, "k"'] & & B \arrow[ul, "k"'] \end{tikzcd} \;\middle|\; k \in \mathbb{R} \right\}$

Which vectors have <u>no</u> cyclic component? Those perpendicular to C: $(a,b,c) \perp (x,y,z) \Longleftrightarrow ax+by+cz=0$.

So $C^{\perp} = \{(a,b,c) \in \mathbb{R}^3 \mid ak+bk+ck=0 \;\; \forall k \in \mathbb{R}\}$

$= \{(a,b,c) \in \mathbb{R}^3 \mid a+b+c=0\}$

$= \{ b(-1,1,0) + c(-1,0,1) \mid b,c \in \mathbb{R}\}$.

This led to the ordered basis $\mathcal{B} = \{(1,1,1), (-1,1,0), (-1,0,1)\}$ of $\mathbb{R}^3$.

~~Ex~~ First coord: cyclic component

2nd, 3rd coords: non-cyclic components.

e.g. $(1,1,-1) = a(1,1,1) + b(-1,1,0) + c(-1,0,1)$

$$\Leftrightarrow \quad \begin{aligned} a-b-c &= 1 \\ a+b &= 1 \\ a+c &= -1 \end{aligned} \quad \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & -4/3 \end{array}\right)$$

so $(1,1,-1)$ has coords $(1/3, 2/3, -4/3)$ wrt $\mathcal{B}$.

In particular, this "rational preference" (i.e. ordered preference) has a cyclic component!

$C > B > A$ has coords $\left(-\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}\right)$ wrt $\mathcal{B}$

Call sign of first coord the <u>spin</u> of the rational preference.

<u>pos spin</u> | <u>neg spin</u>

①

$$\begin{tikzcd}[sep=small] & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "1"'] \end{tikzcd} = \begin{tikzcd}[sep=small] & A \arrow[dl, "1/3"'] & \\ C \arrow[rr, "1/3"'] & & B \arrow[ul, "1/3"'] \end{tikzcd} + \begin{tikzcd}[sep=small] & A \arrow[dl, "-4/3"'] & \\ C \arrow[rr, "2/3"'] & & B \arrow[ul, "2/3"'] \end{tikzcd}$$

$A > B > C$

$$\begin{tikzcd}[sep=small] & A \arrow[dl, "1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "-1"'] \end{tikzcd} = \begin{tikzcd}[sep=small] & A \arrow[dl, "-1/3"'] & \\ C \arrow[rr, "-1/3"'] & & B \arrow[ul, "-1/3"'] \end{tikzcd} + \begin{tikzcd}[sep=small] & A \arrow[dl, "4/3"'] & \\ C \arrow[rr, "-2/3"'] & & B \arrow[ul, "-2/3"'] \end{tikzcd}$$

$C > B > A$

②

$$\begin{tikzcd}[sep=small] & A \arrow[dl, "1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "-1"'] \end{tikzcd} = \begin{tikzcd}[sep=small] & A \arrow[dl, "1/3"'] & \\ C \arrow[rr, "1/3"'] & & B \arrow[ul, "1/3"'] \end{tikzcd} + \begin{tikzcd}[sep=small] & A \arrow[dl, "2/3"'] & \\ C \arrow[rr, "2/3"'] & & B \arrow[ul, "-4/3"'] \end{tikzcd}$$

$B > C > A$

$$\begin{tikzcd}[sep=small] & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "1"'] \end{tikzcd} = \begin{tikzcd}[sep=small] & A \arrow[dl, "-1/3"'] & \\ C \arrow[rr, "-1/3"'] & & B \arrow[ul, "-1/3"'] \end{tikzcd} + \begin{tikzcd}[sep=small] & A \arrow[dl, "-2/3"'] & \\ C \arrow[rr, "-2/3"'] & & B \arrow[ul, "4/3"'] \end{tikzcd}$$

$A > C > B$

③

$$\begin{tikzcd}[sep=small] & A \arrow[dl, "1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "1"'] \end{tikzcd} = \begin{tikzcd}[sep=small] & A \arrow[dl, "1/3"'] & \\ C \arrow[rr, "1/3"'] & & B \arrow[ul, "1/3"'] \end{tikzcd} + \begin{tikzcd}[sep=small] & A \arrow[dl, "2/3"'] & \\ C \arrow[rr, "-4/3"'] & & B \arrow[ul, "2/3"'] \end{tikzcd}$$

$C > A > B$

$$\begin{tikzcd}[sep=small] & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "-1"'] \end{tikzcd} = \begin{tikzcd}[sep=small] & A \arrow[dl, "-1/3"'] & \\ C \arrow[rr, "-1/3"'] & & B \arrow[ul, "-1/3"'] \end{tikzcd} + \begin{tikzcd}[sep=small] & A \arrow[dl, "-2/3"'] & \\ C \arrow[rr, "4/3"'] & & B \arrow[ul, "-2/3"'] \end{tikzcd}$$

$B > A > C$

Summing row ① contributions from election, get $\begin{tikzcd}[sep=small] & A \arrow[dl, "-a"'] & \\ C \arrow[rr, "a"'] & & B \arrow[ul, "a"'] \end{tikzcd}$ with $a > 0$ if more on left, $a < 0$ if more on right, $a = 0$ if same left/right.

From row ②, sim get $\begin{tikzcd}[sep=small] & A \arrow[dl, "b"'] & \\ C \arrow[rr, "b"'] & & B \arrow[ul, "-b"'] \end{tikzcd}$, and from ③ $\begin{tikzcd}[sep=small] & A \arrow[dl, "c"'] & \\ C \arrow[rr, "-c"'] & & B \arrow[ul, "c"'] \end{tikzcd}$.

The election is then determined by

$$\begin{tikzcd} & A \arrow[dl, "-a+b+c"'] & \\ C \arrow[rr, "a+b-c"'] & & B \arrow[ul, "a-b+c"'] \end{tikzcd}$$

Condorcet cycle when all 3 have same sign.

All positive:

$$\left.\begin{array}{l} -a+b+c>0 \\ a-b+c>0 \\ a+b-c>0 \end{array}\right\} \underset{\text{check}}{\Longrightarrow} \quad a,b,c>0$$

Thus we have proved the following:

<u>Thm</u> If there is a Condorcet cycle, then $a,b,c>0$ or $a,b,c<0$.

<u>TPS</u> Converse?

<u>Q</u> Given N voters, what fraction of voting profiles result in Condorcet cycles?