Today:
- Compute reduced echelon form of an augmented matrix.
- Learn how to express an infinite number of solutions in parametric and vector forms.

In a matrix, the <u>leading term</u> of a row is its first nonzero entry. A matrix is in <u>echelon form</u> if each leading term is to the right of the leading term in the row above it (except for the leading term in the first row) and any all 0 rows are at the bottom:

A matrix is in <u>reduced echelon form</u> if it is in echelon form and each leading term is a 1 and is the only nonzero entry in its column.

<u>e.g.</u>
$$\left(\begin{array}{cccc|c} 1 & & & & a \\ & 1 & & & b \\ & & 1 & & c \\ & & & 1 & d \end{array}\right) \Rightarrow \begin{array}{l} x_1 = a \\ x_2 = b \\ x_3 = c \\ x_4 = d \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 2 & & 3 & a \\ & & 1 & 4 & b \\ & & & 1 & c \\ & & & & d \end{array}\right) \Rightarrow \text{? (TPS)}$$

<u>TPS</u>
- When are there no solutions? — contradictory eq'n
- When is there a unique solution? — no contradiction, every column has a leading term

In reduced echelon form:

<u>e.g.</u>
$$\begin{aligned} 2x_3 + 6x_4 &= 0 \\ x_1 + 2x_2 + x_3 + 3x_4 &= 1 \\ 2x_1 + 4x_2 + 3x_3 + 9x_4 + x_5 &= 5 \end{aligned}$$
has augmented matrix

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 6 & 0 & 0 \\ 1 & 2 & 1 & 3 & 0 & 1 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right) \xrightarrow{r_3 \to r_3 - 2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}\right)$$

$$\xrightarrow[r_2 \to \frac{1}{2} r_2]{} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_2]{r_1 \to r_1 - r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

x_2, x_4 are the <u>free</u> variables.

So the original system is equivalent to

$$\begin{aligned} x_1 + 2x_2 \qquad\qquad &= 1 \quad \Rightarrow \quad x_1 = 1 - 2x_2 \\ x_3 + 3x_4 \quad &= 0 \quad \Rightarrow \quad x_3 = -3x_4 \\ x_5 &= 3 \quad \Rightarrow \quad x_5 = 3 \end{aligned}$$

Solution set: $\{(1-2x_2,\ x_2,\ -3x_4,\ x_4,\ 3) \mid x_2, x_4 \in \mathbb{R}\}$

a plane of solutions. This is a <u>parametric</u> description of the sol'ns.

Vector form: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$

<u>e.g.</u> If a system has reduced echelon form

$$\left(\begin{array}{ccccccc|c} 1 & 3 & 0 & 1 & 2 & 0 & 1 & 7 \\ 0 & 0 & 1 & 4 & -1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

then its solution set is

$x_2 \quad x_4 \quad x_5 \quad x_7$ free

$\{(7-3x_2-x_4-2x_5-x_7,\ x_2,\ -2-4x_4+x_5-3x_7,\ x_4,\ x_5,\ 3-x_7,\ x_7) \mid x_2, x_4, x_5, x_7 \in \mathbb{R}\}$

(parametric form)

or $\left\{ \begin{pmatrix} 7 \\ 0 \\ -2 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_4, x_5, x_7 \in \mathbb{R} \right\}$

(vector form).

<u>Problem</u> Find all parabolas $y = ax^2 + bx + c$ passing through $(1,4)$ & $(3,6)$.

<u>Sol'n</u> To pass through $(1,4)$ we need

$$4 = a + b + c.$$

For $(3,6)$,

$$6 = 9a + 3b + c.$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & 3 & 1 & 6 \end{array}\right) \xrightarrow{r_2 \to r_2 - 9r_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & -8 & -30 \end{array}\right) \xrightarrow{r_2 \to \frac{-1}{6} r_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 4/3 & 5 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1/3 & -1 \\ 0 & 1 & 4/3 & 5 \end{array}\right)$$ in reduced echelon form. Thus

the parabolas in question have

$$(a,b,c) \in \left\{ \left(-1 + \tfrac{1}{3}c,\ 5 - \tfrac{4}{3}c,\ c\right) \,\middle|\, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1/3 \\ -4/3 \\ 1 \end{pmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$

<u>TPS</u> Check this.