Linear algebra is <u>pervasive</u> in modern math/science/tech:

- quantum physics
- Google PageRank
- machine learning (PCA, etc.)
- Markov processes
- multivariable differentiation
- multidimensional volume

...

But its origins are <u>elementary</u>:

<u>e.g.</u> Find all (x,y) such that

$$3x+2y=5$$
$$2x-y=1.$$

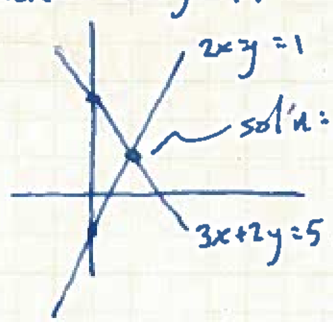
Eliminate variables:

$$\begin{array}{rl} 3x+2y &= 5 \\ 4x-2y &= 2 \\ \hline 7x &= 7 \end{array} \Rightarrow x=1 \text{ and } 3x+2y=5 \text{ so}$$

$$3+2y=5$$
$$\Rightarrow 2y=2$$
$$\Rightarrow y=1$$

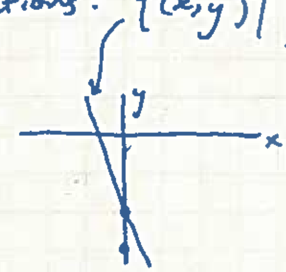
Unique solution: $x=y=1$.

Geometry:



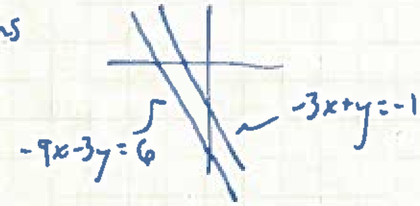
<u>e.g.</u>
$$-9x-3y=6$$
$$3x+y=-2$$

Solutions: $\{(x,y) \mid y=-2-3x\}$



e.g. $-9x - 3y = 6$
$3x + y = -1$

No solutions



e.g.
$$\begin{aligned} x + 2y + z &= 0 \\ x \quad + z &= 4 \\ x + y + 2z &= 1 \end{aligned}$$

General idea: Replace a given set of equations with an equivalent set (having the same solution set) but from which solutions are evident.

The following operations do not change the solution set and are called row operations:

1. Multiply an equation by a nonzero scalar
2. Swap two equations
3. Add a multiple of one row to another

scalar ← element of the "base field" F (maybe $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Z}/5\mathbb{Z}$)

Think Pair Share Why are these operations invertible? Why does this imply solution sets are invariant under row operations?

We will see that row operations are sufficient to solve our problem.

$$\begin{aligned} x + 2y + z &= 0 \\ x \quad + z &= 4 \\ x + y + 2z &= 1 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[\substack{r_3 \to r_3 - r_1}]{r_2 \to r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

(eliminate x from last two eq'ns)

- augmented matrix
- columns correspond to coefficient of x, y, z, and constant value

$$\xrightarrow{r_2 \to -\frac{1}{2} r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{r_3 \to r_3 + r_2} \underbrace{\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)}_{\text{echelon form}} \xrightarrow{r_1 \to -2r_2 + r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right),$$

(set coeff of y in 2nd eqn to 1)

$$\xrightarrow{r_1 \to r_1 - r_3} \underbrace{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)}_{\text{reduced echelon form}} \rightsquigarrow \left.\begin{aligned} x &= 5 \\ y &= -2 \\ z &= -1 \end{aligned}\right\}$$

Unique solution! Check that it works.

e.g.

$$\begin{aligned} x + 2y + z &= 0 \\ x \quad + z &= 4 \\ x + y + z &= 1 \end{aligned} \quad \text{i.e.} \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

No solutions as the final row says that $0 = -1$!

e.g.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{aligned} x + z &= 4 \\ y &= -2 \\ (0 &= 0) \end{aligned}$$

Solution set: $\{(x, -2, 4-x) \mid x \in \mathbb{R}\}$, a line in $\mathbb{R}^3$