Vector Spaces

Let F be a field, e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/2\mathbb{Z}$, etc. (not $\mathbb{Z}$...).

Defn A vector space over F (or F-vector space) is a set V together with operations $+ : V \times V \longrightarrow V$ (vector addition)

$\cdot : F \times V \longrightarrow V$ (scalar multiplication)

(Write $v+w$ for $+(v,w)$, λv for $\cdot(\lambda, v)$.) These operations have the following properties for all $x,y,z \in V$, $a,b \in F$:

1. $x+y = y+x$ (commutativity of $+$)
2. $(x+y)+z = x+(y+z)$ (associativity of $+$)
3. $\exists\, 0 \in V$ s.t. $w + 0 = w$ $\forall w \in V$
4. $\exists -x \in V$ s.t. $x + (-x) = 0$
5. For $1 \in F$, $1 \cdot x = x$
6. $(ab)x = a(bx)$ (associativity of scalar mult)
7. $a(x+y) = ax + ay$
8. $(a+b)x = ax + bx$

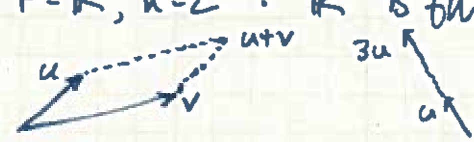
(7)–(8): distributivity

Rmk (2)-(4) make V a group under $+$. (1) makes this group Abelian. (5)-(8) say that F acts on V in a manner compatible with $+$. All together, we get a linear structure on V.

e.g. $F^n = \underbrace{F \times \cdots \times F}_{n \text{ factors}} = \{(a_1, \dots, a_n) \mid a_i \in F \text{ for } i=1,\dots,n\}$

$$(a_1,\dots,a_n) + (b_1,\dots,b_n) := (a_1+b_1, \dots, a_n+b_n)$$
$$c(a_1,\dots,a_n) := (ca_1, \dots, ca_n)$$

sub-eg. (a) $F = \mathbb{R}$, $n=2$: $\mathbb{R}^2$ is the Euclidean plane



(b) $F = \mathbb{Z}/2\mathbb{Z}$, $n=3$: vector space with 8 elts such as $(0,1,0)$, $(0,1,1)$ with $(0,1,0)+(0,1,1) = (0,0,1)$.

ⓒ $n=1$: $F^1 = F$

ⓓ $n=0$: $F^0 = \{0\}$, the trivial vector space.

$$0+0=0$$
$$a0=0$$

e.g. $\mathbb{C}$ is an $\mathbb{R}$-vector space :

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$
$$c(a+bi) = ca + cbi$$

e.g. $\mathbb{R}$ is a $\mathbb{Q}$-vector space.

e.g. $M_{m\times n}(F)$ = $m\times n$ matrices with entries in F

$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \middle| \ a_{ij} \in F \ \forall i,j \right\}$$

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

ij entry of A

$$(cA)_{ij} = c(A_{ij})$$

Q How similar is this to F^{mn} ?

e.g. For S any set, let $F^S := \{\text{functions } f: S \to F\}$.

$$(f+g)(s) = f(s)+g(s)$$
$$(cf)(s) = c(f(s))$$

If $S=\{1,\dots,n\}$, F^S is essentially the same as F^n :

$$f:\{1,\dots,n\} \to F \iff (f(1),\dots,f(n)) \in F^n$$

If $S=\mathbb{N}$, get sequences in F.

TPS Put a linear structure on the set of polynomials in a single variable.