The Spectral Thm

Defn Suppose $(V, \langle , \rangle)$ a real or complex inner product space. A linear transformation $S \in \mathcal{L}(V)$ is self-adjoint if

$$\langle Sv, w \rangle = \langle v, Sw \rangle \quad \forall v, w \in V$$

A square matrix A is self-adjoint if $A^{+} = A$, which is equivalent to the associated linear trans being self-adjoint, and vice versa.

Note If $F = \mathbb{R}$, then self-adjoint = symmetric.

Spectral Thm Suppose V is a finite dimensional real or cpx inner product space. Then $S \in \mathcal{L}(V)$ is self-adjoint iff $V = E_{\lambda_1}(S) \oplus \cdots \oplus E_{\lambda_m}(S)$ and $E_{\lambda_i}(S) \subseteq E_{\lambda_j}(S)^{\perp}$ for $i \neq j$ where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of S and $E_{\lambda_i}(S) = \ker(S - \lambda_i I)$ is the λ_i-eigenspace of S. Moreover, each $\lambda_i \in \mathbb{R}$.
(This is called an orthogonal direct sum decomposition of V. Exc Concatenating orthonormal bases of the $E_{\lambda_i}(S)$'s results in an orthonormal basis of eigenvectors of S for V.)

Lemma 1 Suppose $S \in \mathcal{L}(V)$ is self-adjoint and $U \subseteq V$ subspace satisfies $S(U) \subseteq U$ (so U is "S-invariant"). Then $U^{\perp}$ is S-invariant as well: $S(U^{\perp}) \subseteq U^{\perp}$.

Pf For $u \in U$ and $v \in U^{\perp}$, $\langle u, S(v) \rangle = \langle Su, v \rangle$. Since $Su \in U$ and $v \in U^{\perp}$, this quantity is 0, so $S(v) \in U^{\perp}$. $\square$

Lemma 2 Suppose $S \in \mathcal{L}(V)$ is self-adjoint on a finite-dim'l inner prod space V. The function $s: v \in V \longmapsto \langle Sv, v \rangle$ takes real values. If v_0 is a maximum for s on the unit sphere

$$S_V := \{ v \in V \mid \langle v, v \rangle = 1 \}$$

then v_0 is an eigenvector of S with (real) eigenvalue $s(v_0)$

<u>Pf of L2</u> Later. $\square$

Assuming L2, we give a

<u>Pf of the Spectral Thm</u> ($\Leftarrow$) Exc. (Easy.)

($\Rightarrow$) Proceed by induction on $\dim V$. If $\dim V = 0$, then $S = 0$ and there are no eigenvalues. $\{0\}$ is an empty direct sum (i.e. has basis $\varnothing$), so done.

Suppose $\dim V > 0$ and that the theorem is known for spaces of lower dimension. Since $V \neq 0$, $\text{sphere}(V) \neq \varnothing$. The function s is continuous (<u>exc</u> upon choosing a basis for V, s is a quadratic polynomial fn of the coords).

<u>Topological fact</u> Any cts fn on a closed bounded set attains a maximum $s(v_0)$ at some pt v_0.

Big deal theorem!

By Lemma 2, $v_0 \in E_{s(v_0)}(S)$, so the eigenspace is not trivial. Moreover, $E_{s(v_0)}(S)$ is S-invariant, so Lemma 1 implies $V = E_{s(v_0)}(S) \oplus E_{s(v_0)}(S)^\perp$, an orthogonal direct sum of S-invariant spaces. Now

$$\dim E_{s(v_0)}(S)^\perp = \dim V - \dim E_{s(v_0)}(S) < \dim V.$$

By the inductive hypothesis, $E_{s(v_0)}(S)^\perp$ is an orthogonal direct sum of eigenspaces for real eigenvalues of (the restriction of) S. $\square$

<u>Pf of Lemma 2</u> Suppose v_0 a maximum for s on $\text{sphere}(V)$.

<u>Claim</u> if $\langle v_0, u\rangle = 0$, then $\text{Re}\langle S(v_0), u\rangle = 0$.

Assuming the claim, get that

$$\text{if } \langle v_0, u\rangle = 0, \text{ then } \langle S(v_0), u\rangle = 0$$

by applying the claim to $e^{i\theta}u$ and using sesquilinearity.

Thus $u \in \operatorname{span}\{v_0\}^\perp \Rightarrow \langle u, S(v_0)\rangle = 0$.
so $S(v_0) \in (\operatorname{span}\{v_0\}^\perp)^\perp = \operatorname{span}\{v_0\}$, in which case $S(v_0) = \lambda v_0$ for some $\lambda \in F$. We have

$$\lambda = \lambda \frac{\langle v_0, v_0\rangle}{\langle v_0, v_0\rangle} = \langle S(v_0), v_0\rangle = s(v_0), \text{ as desired.}$$

It remains to prove the claim. By sesquilinearity, it suffices to consider u of ~~...~~ length 1. Then have that u, v_0 are orthonormal. By Pythagoras,

$$v_t = \cos(t) v_0 + \sin(t) u \qquad (t \in \mathbb{R})$$

are all unit vectors. (Note $v_0 = v_0$).

Thus $f : \mathbb{R} \longrightarrow \mathbb{R}$, $t \longmapsto s(v_t) = \langle S(v_t), v_t\rangle$ has maximum at $t = 0$.

We have $f(t) = \langle Sv_t, v_t\rangle = \langle \cos t\, Sv_0 + \sin t\, Su, \cos t\, v_0 + \sin t\, u\rangle$

$$= \cos^2 t \langle Sv_0, v_0\rangle + \sin^2 t \langle Su, u\rangle + \cos t \sin t\,(\langle Sv_0, u\rangle + \langle Su, v_0\rangle)$$

By self-adjointness of S, $\langle Su, v_0\rangle = \overline{\langle Sv_0, u\rangle}$, whence

$$f(t) = \cos^2 t \langle Sv_0, v_0\rangle + \sin^2 t \langle Su, u\rangle + 2\cos t \sin t \operatorname{Re}\langle Sv_0, u\rangle$$

Thus f is diff'l and

$$f'(0) = 2 \operatorname{Re}\langle Sv_0, u\rangle.$$

Since 0 is a max, $f'(0) = 0 \Rightarrow \operatorname{Re}\langle Sv_0, u\rangle = 0$. $\square$