Orthogonal complements and projections

Defn The (external) direct sum of vector spaces U, W over a field F is the set $U \oplus W := U \times W$ with coordinatewise scalar mult & vector addition:

$$\lambda(u,w) = (\lambda u, \lambda w)$$
$$(u,w) + (u',w') = (u+u', w+w') \qquad u,u' \in U,\ w,w' \in W,\ \lambda \in F$$

Prop Let U, W be subspaces of a vector space V over F s.t. $U + W = V$ and $U \cap W = \{0\}$. Then

$$U \oplus W \xrightarrow{\cong} V$$
$$(u,w) \longmapsto u+w. \qquad \square$$

~~Def~~ Now let $(V, \langle , \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

Defn Let $\emptyset \neq S \subseteq V$. The orthogonal complement of S is

$$S^\perp = \{x \in V \mid \langle x, y \rangle = 0 \ \forall y \in S\}.$$

Exc $S^\perp$ is a subspace of V.

Prop Suppose $\dim V = n$ and $S = \{v_1, \dots, v_k\}$ is an orthonormal subset of V. ① S can be extended to an orthonormal basis $\{v_1, \dots, v_n\}$ of V.

② If $W = \operatorname{span} S$, then $S' = \{v_{k+1}, \dots, v_n\}$ is an orthonormal basis for $W^\perp$.

③ If $W \subseteq V$ is any subspace, then

$$\dim W + \dim W^\perp = \dim V = n.$$

④ If $W \subseteq V$ is any subspace, then $(W^\perp)^\perp = W$.

Pf ① Extend to a basis then apply G-S.

② S' is lin ind as it's a subset of a basis. $S' \subseteq W^\perp$ by orthogonality of $\{v_1, \dots, v_n\}$. Thus $\operatorname{span} S' \subseteq W^\perp$.

For $x \in W^\perp$, $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i = \sum_{i=k+1} \langle x, v_i \rangle v_i \in \operatorname{span} S'$.

③ Choose an orthonormal basis for W then apply ①, ②.

④ Have $(W^\perp)^\perp = \{x \in V \mid \langle x, y\rangle = 0 \ \forall y \in W^\perp\} \supseteq W$.

By ③, $\dim (W^\perp)^\perp = n - \dim W^\perp = \dim W$, so they are equal. □

<u>Prop</u> Let W be a finite dimensional subspace of V. Then $V = W \oplus W^\perp$. I.e. $\forall y \in V \ \exists!\, u \in W, z \in W^\perp$ s.t. $y = u + z$.

<u>Defn</u> Define u to be the <u>orthogonal projection</u> of y onto W.

If $u_1, \ldots, u_k$ <u>orthonormal</u> basis of W, then

$$u = \sum_{i=1}^{k} \langle y, u_i\rangle u_i .$$

<u>Pf</u> By G-S, $\exists$ orthonormal basis $u_1, \ldots, u_k$ of W. Define $u = \sum_{i=1}^{k} \langle y, u_i \rangle u_i$ and $z = y - u$. Then $u \in W$ and $y = u + z$ for $z = y - u$. Further, for $j = 1, \ldots, k$,

$$\begin{aligned}
\langle z, u_j\rangle &= \langle y - u, u_j\rangle \\
&= \langle y, u_j\rangle - \left\langle \sum_{i=1}^{k} \langle y, u_i\rangle u_i, u_j \right\rangle \\
&= \langle y, u_j\rangle - \sum_{i=1}^{k} \langle y, u_i\rangle \langle u_i, u_j\rangle \\
&= \langle y, u_j\rangle - \langle y, u_j\rangle \langle u_j, u_j\rangle \\
&= 0, \text{ so } z \in W^\perp.
\end{aligned}$$

For uniqueness, suppose $\exists u' \in W, z' \in W^\perp$ s.t. $y = u + z = u' + z'$. Then $u - u' = z - z' \in W \cap W^\perp = \{0\}$. □

<u>Cor</u> The orthogonal projection u of y onto W is the closest vector in W to y: $\|y - u\| \le \|y - w\| \ \forall w \in W$ with equality iff $u = w$.

<u>Pf</u> Write $y = u + z$ with $u \in W, z \in W^\perp$. Let $w \in W$. Then $u - w \in W$, $y - u \in W^\perp$. So $u - w$ and $z = y - u$ are orthogonal.

By Pythagoras,

$$\begin{aligned}\|y-w\|^2 &= \|(u+z)-w\|^2 \\ &= \|(u-w)+z\|^2 \\ &= \|u-w\|^2 + \|z\|^2 \\ &\geq \|z\|^2 \\ &= \|y-u\|^2.\end{aligned}$$

Equality iff $\|u-w\|^2 = 0$, i.e. iff $u=w$. $\square$

e.g. $V=\mathbb{R}^3$ w/ std inner prod. For orthogonal proj'n onto xy-plane, take $\{e_1, e_2\}$ as orthonormal basis. The proj'n of $u=(x,y,z)\in\mathbb{R}^3$ onto this plane is

$$(u\cdot e_1)e_1 + (u\cdot e_2)e_2 = (x,y,0).$$

The distance of u to the xy-plane is

$\cancel{\|u-(0,0,z}$

$$\|u-(x,y,0)\| = \|(0,0,z)\| = |z|. \checkmark$$