Gram-Schmidt

Let $(V, \langle , \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

Defn Let $S \subseteq V$. Then S is an orthogonal subset of V if $\langle u, v\rangle = 0$ for all $u \neq v \in S$. If S is an orthogonal subset of V and $\|u\| = 1$ for all $u \in S$, then S is an orthonormal subset of V.

e.g. • The standard basis $e_1, \dots, e_n$ for F^n is orthonormal with respect to the standard inner product on F^n.

• $\{\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)\}$ is orthonormal wrt std inner product on $\mathbb{R}^2$

Prop Let $S = \{v_1, \dots, v_k\}$ be an orthogonal set of nonzero vectors in V, and let $y \in \operatorname{span} S$. Then

$$y = \sum_{j=1}^{k} \frac{\langle y, v_j\rangle}{\langle v_j, v_j\rangle} v_j = \sum_{j=1}^{k} \frac{\langle y, v_j\rangle}{\|v_j\|^2} v_j .$$

Pf Say $y = \sum_{i=1}^{k} a_i v_i$. Then for $j = 1, \dots, k$

$$\langle y, v_j\rangle = \left\langle \sum_{i=1}^{k} a_i v_i , v_j \right\rangle = \sum_{i=1}^{k} a_i \langle v_i, v_j\rangle = a_j \langle v_j, v_j\rangle .$$

Hence $a_j = \dfrac{\langle y, v_j\rangle}{\langle v_j, v_j\rangle}$. $\square$

Cor Let $S \subseteq V$ be orthonormal, $S = \{v_1, \dots, v_k\}$, $y \in \operatorname{span} S$. Then $y = \sum_{j=1}^{k} \langle y, v_j\rangle v_j$. $\square$

Cor If $S = \{v_1, \dots, v_k\} \subseteq V$ is orthogonal, then S is linearly ind.

Pf If $\sum_{i=1}^{k} a_i v_i = 0$ then for $j = 1, \dots, k$,

$$0 = \langle 0, v_j\rangle = \left\langle \sum_{i=1}^{k} a_i v_i , v_j \right\rangle = a_j \underbrace{\langle v_j, v_j\rangle}_{\neq 0} \Rightarrow a_j = 0 \ \forall j. \ \square$$

e.g. $\mathbb{R}^2$ w/ std inner prod,

$$u = \frac{1}{\sqrt{2}}(1,1), \quad v = \frac{1}{\sqrt{2}}(1,-1)$$

Then $\beta = \{u, v\}$ is an orthonormal ordered basis for $\mathbb{R}^2$.

Q What are the coords of $y = (4,7)$ wrt β?

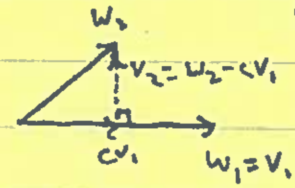
A (TPS)
$$\begin{aligned} y &= \langle y,u\rangle u + \langle y,v\rangle v \\ &= (4,7)\cdot\tfrac{1}{\sqrt{2}}(1,1)\, u + (4,7)\cdot\tfrac{1}{\sqrt{2}}(1,-1)\, v \\ &= \frac{11}{\sqrt{2}} u - \frac{3}{\sqrt{2}} v \end{aligned}$$

Indeed, $\frac{11}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(1,1) - \frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}}(1,-1) = \left(\frac{11}{2},\frac{11}{2}\right) - \left(\frac{3}{2},-\frac{3}{2}\right) = (4,7)$ ✓

Gram-Schmidt

Baby case: given $w_1, w_2 \in V$, find orthogonal v_1, v_2 s.t.
$$\operatorname{span}\{w_1,w_2\} = \operatorname{span}\{v_1,v_2\}.$$

Idea: Let $v_1 = w_1$, then "straighten out" w_2 to create v_2:



Here $c = \dfrac{\langle w_2, v_1\rangle}{\|v_1\|^2}$ is the component of w_2 along v_1.

Gram-Schmidt Orthogonalization Algorithm:

Input: $S = \{w_1, \dots, w_n\}$, a lin ind subset of V.

- Let $v_1 = w_1$.
- For $k = 2, 3, \dots, n$, define
$$v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i\rangle}{\|v_i\|^2} v_i .$$

(I.e. subtract orthog proj's of w_k along previously found v_j.)

Output: $S' = \{v_1, \dots, v_n\}$ an orthogonal set with span S' = span S

or Output: $S'' = \left\{\frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|}\right\}$ an orthonormal set with span S'' = span S.

Pf by induction on n. For $n=1$, ✓. Assume it works for some $n \geq 1$. Then $\operatorname{span}\{v_1, \ldots, v_n\} = \operatorname{span}\{w_1, \ldots, w_n\}$ and $\{v_1, \ldots, v_n\}$ orthogonal. Then

$$v_{n+1} = w_{n+1} - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i .$$

If $v_{n+1} = 0$, then $w_{n+1} \in \operatorname{span}\{v_1, \ldots, v_n\} = \operatorname{span}\{w_1, \ldots, w_n\}$, ※, so $v_{n+1} \neq 0$. For $j = 1, \ldots, n$,

$$\langle v_{n+1}, v_j \rangle = \left\langle w_{n+1} - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i , v_j \right\rangle$$

$$= \langle w_{n+1}, v_j \rangle - \sum_{i=1}^{n} \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} \langle v_i, v_j \rangle$$

$$= \langle w_{n+1}, v_j \rangle - \frac{\langle w_{n+1}, v_j \rangle}{\|v_j\|^2} \langle v_j, v_j \rangle$$

$$= 0.$$

so $\{v_1, \ldots, v_{n+1}\}$ orthogonal. It remains to show this set has the correct span. Since $\{v_1, \ldots, v_{n+1}\}$ lin ind and

$$\operatorname{span}\{v_1, \ldots, v_{n+1}\} \subseteq \operatorname{span}\{v_1, \ldots, v_n, w_{n+1}\} \subseteq \operatorname{span}\{w_1, \ldots, w_{n+1}\}$$

get equality (both $(n+1)$-dim'l). $\square$

Cor Every nontrivial fin dim'l inner product space has an orthonormal basis. $\square$

e.g. $V = \mathbb{R}_{\leq 1}[x]$, $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$.

Apply G-S to $\{1, x\}$ to get orthonormal basis.

Note $\langle 1, x \rangle = \int_0^1 t\, dt = \frac{1}{2} \neq 0$, so not orthogonal.

G-S: $v_1 = 1$.

$$v_2 = x - \frac{\langle x, v_1 \rangle}{\|v_1\|^2} v_1 = x - \frac{\langle x, 1 \rangle}{\|1\|^2} 1 = x - \frac{1/2}{\int_0^1 dt} = x - \frac{1}{2}.$$

Have $\|v_1\| = \sqrt{\int_0^1 dt} = 1$

$$\|v_2\| = \sqrt{\langle x - 1/2, x - 1/2 \rangle}$$
$$= \sqrt{\int_0^1 (t - 1/2)^2 \, dt}$$
$$= \sqrt{1/12}$$

So an orthonormal basis for V is $\left\{1, \frac{1}{\sqrt{12}}(x - 1/2)\right\}$.