Inner Products

Goal: Add structure to a vector space that will allow us to define length and angles.

Defn Let V be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$. An inner product on V is a function

$$\langle \, , \, \rangle : V \times V \longrightarrow F$$
$$(x,y) \longmapsto \langle x, y \rangle$$

s.t. $\forall x,y,z \in V$, $c \in F$,

(1) $\langle x+y, z \rangle = \langle x,z \rangle + \langle y,z \rangle$ and $\langle cx, y \rangle = c\langle x,y \rangle$

(2) $\overline{\langle x,y \rangle} = \langle y,x \rangle$

(3) $\langle x,x \rangle \in \mathbb{R}_{\geq 0}$ and $\langle x,x \rangle = 0$ iff $x=0$.

Note $F = \mathbb{R}$: non-degenerate positive definite form

$F = \mathbb{C}$: non-degenerate Hermitian form

e.g.
- The ordinary dot product on $\mathbb{R}^n$: $V = \mathbb{R}^n$,
$$\langle (x_1,\dots,x_n), (y_1,\dots,y_n) \rangle = x \cdot y = \sum_{i=1}^n x_i y_i$$
- The ordinary inner product on $\mathbb{C}^n$: $V = \mathbb{C}^n$
$$\langle (x_1,\dots,x_n), (y_1,\dots,y_n) \rangle = x \cdot \bar{y} = \sum_{i=1}^n x_i \bar{y}_i \, .$$
- Let $V = C_{\mathbb{R}}([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid f \text{ cts} \}$,
$$\langle f, g \rangle = \int_0^1 f(t) g(t) \, dt$$

TPS Check pos def.

- $V = \mathbb{R}^2$, $\langle (x_1,x_2), (y_1,y_2) \rangle = 3x_1y_1 + 2x_1y_2 + 2x_2y_1 + 4x_2y_2$

For pos def, $\langle (x_1,x_2), (x_1,x_2) \rangle = 3x_1^2 + 4x_1x_2 + 4x_2^2$

$$= 3\left(x_1^2 + \tfrac{4}{3}x_1x_2 + \tfrac{4}{3}x_2^2\right)$$
$$= 3\left(\left(x_1 + \tfrac{2}{3}x_2\right)^2 - \tfrac{4}{9}x_2^2 + \tfrac{4}{3}x_2^2\right)$$
$$= 3\left(\left(x_1 + \tfrac{2}{3}x_2\right)^2 + \tfrac{8}{9}x_2^2\right) \geq 0$$

with equality iff $x_1 = x_2 = 0$.

- $V = M_{m \times n}(F)$. For $A \in V$, define the <u>conjugate transpose</u> of A by $A^* := \bar{A}^T$ where $\bar{(\ )}$ takes the cpx conjugate of each entry of A. Define

$$\langle A, B \rangle = \operatorname{tr}(B^* A) = \sum_{i=1}^{n} (B^* A)_{ii}$$

<u>Note</u> $m = 1$ gives usual inner product

Pos def: exercise.

<u>Prop</u> Let $(V, \langle , \rangle)$ be an inner product space. Then

(1) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(2) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$

(3) $\langle 0, y \rangle = 0$

(4) if $\langle x, y \rangle = \langle x, z \rangle \ \forall x \in V$ then $y = z$.

<u>Pf</u> (1): $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle}$

$$= \langle x, y \rangle + \langle x, z \rangle$$

(2), (3): exc.

(4): $\langle x, y \rangle = \langle x, z \rangle \ \forall x \Rightarrow \langle x, y - z \rangle = 0 \ \forall x$

$\cancel{\Rightarrow y - z = 0}$

In particular, for $x = y - z$ get $\langle y-z, y-z \rangle = 0$, so $y - z = 0$.

$\square$