Suppose $x_1(t)$ = pop'n of frogs in a pond
$x_2(t)$ = pop'n of flies in a pond

and suppose

$$(*) \quad \begin{aligned} x_1'(t) &= a x_1(t) + b x_2(t) \\ x_2'(t) &= c x_1(t) + d x_2(t) \end{aligned}$$

Let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$

Then $(*) \iff (**) \quad x'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x(t)$

<u>Goal</u> Find $x(t)$ solving $(**)$.

<u>e.g.</u> If $b=c=0$, get $x_1'(t) = a x_1(t)$
$x_2'(t) = d x_1(t)$

so $x_1(t) = k_1 e^{at}$, $x_2(t) = k_2 e^{dt}$, $k_1 = x_1(0)$, $k_2 = x_2(0)$.

Say the system is <u>decoupled</u> since x_1, x_2 don't depend on each other.

Generalize by setting $x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $x' = Ax$ for some $A \in M_{n \times n}(\mathbb{R})$. If A is diagonalizable over $\mathbb{R}$, then decouple the system as follows: take P s.t.

$$P^{-1}AP = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

Then $x' = Ax$ becomes $x' = PDP^{-1}x$

$$\iff P^{-1}x' = DP^{-1}x$$

Set $y(t) = P^{-1}x(t)$. Then $y'(t) = P^{-1}x'(t)$ and we get the system $y' = Dy$, i.e. $y_1' = \lambda_1 y_1$

$$\vdots$$

$$y_n' = \lambda_n y_n$$

Solutions $y_i(t) = k_i e^{\lambda_i t}$ for $k_i = y_i(0)$, $i = 1, \dots, n$.
Since $x = Py$, this solves the original system with a linear combination of $k_1 e^{\lambda_1 t}, \dots, k_n e^{\lambda_n t}$.

e.g. $x_1' = x_2$ i.e. $x' = Ax$ for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
$x_2' = x_1$

Then $P^{-1}AP = D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Thus $y_1 = k_1 e^t$, $y_2 = k_2 e^{-t}$ and $x = Py$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} k_1 e^t + k_2 e^{-t} \\ k_1 e^t - k_2 e^{-t} \end{pmatrix}$$

Suppose the sol'n begins at $(1,0)$. Then

$$1 = x_1(0) = k_1 e^0 + k_2 e^0 = k_1 + k_2$$
$$0 = x_2(0) = k_1 - k_2$$

so $k_1 = k_2 = \frac{1}{2}$ and our sol'n is

$$x_1(t) = \tfrac{1}{2}(e^t + e^{-t})$$
$$x_2(t) = \tfrac{1}{2}(e^t - e^{-t})$$

Note May think of $x' = Ax$ specifying "velocity" x' at each $x \in \mathbb{R}^n$. A sol'n is then $\gamma : \mathbb{R} \to \mathbb{R}^n$ s.t. $\gamma' = A\gamma$ i.e. a "flow" through the velocity field.

Alternate solution
Recall $e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k$ converges $\forall t \in \mathbb{C}$. Given $A \in M_{n\times n}(\mathbb{C})$
define $e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = I_n + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots$.

In each entry we get a power series in t that (Thm) converges for all t.

Prop For $A \in M_{n\times n}(\mathbb{R})$, the sol'n to $x' = Ax$ with initial condition $x(0) = p$ is $x = e^{At}p$.

Sketch $(e^{At})' = Ae^{At} \Rightarrow (e^{At}p)' = A(e^{At}p)$.

Computing e^{At}: If $P^{-1}AP = D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, then $A^k = P\,\mathrm{diag}(\lambda_1^k, \dots, \lambda_n^k)P^{-1}$. Thus

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k = \sum_{k=0}^{\infty} \frac{1}{k!}(PD^kP^{-1})t^k$$

$$= P\left(\sum_{k=0}^{\infty} \frac{1}{k!}D^k t^k\right)P^{-1} = Pe^{Dt}P^{-1}$$

and $e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$

so $e^{At} = P\,\mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})P^{-1}$.

e.g. Previously, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

so $e^{At} = Pe^{Dt}P^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}$$

If $x(0) = (1, 0)$, then $x = e^{At}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{pmatrix}$

as we saw earlier!