Length, distance, components, projections, angles

Defn For $(V, \langle , \rangle)$ an inner product space, the norm (or length) of $x \in V$ is $\|x\| = \sqrt{\langle x, x\rangle} \in \mathbb{R}$. Two vectors are orthogonal (or perpendicular) if $\langle x, y\rangle = 0$. A unit vector x has $\|x\| = 1$ $\iff \langle x, x\rangle = 1$.

e.g. $(\mathbb{R}^n, \cdot)$ has $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$

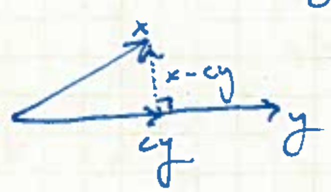
$(\mathbb{C}^n, \cdot)$ has
$$\|z\| = \sqrt{z_1\bar{z}_1 + \cdots + z_n\bar{z}_n} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

Note If $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, then
$$\|z\| = \sqrt{x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2}.$$

Thm (Pythagoras?) Let $(V, \langle , \rangle)$ be an inner product space and suppose $\langle x, y\rangle = 0$. Then $\|x\|^2 + \|y\|^2 = \|x+y\|^2$.

Pf We have $\langle y, x\rangle = \overline{\langle x, y\rangle} = \bar{0} = 0$ as well. Thus
$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2.$$
$\square$

For $x, y \in V$, $x = (x - cy) + cy$, and cy is in the "direction" of y.



Find c s.t. $\langle x - cy, y\rangle = 0$
$$\iff \langle x, y\rangle - c\langle y, y\rangle = 0$$
$$\iff c = \frac{\langle x, y\rangle}{\langle y, y\rangle} = \frac{\langle x, y\rangle}{\|y\|^2}$$
(as long as $y \neq 0$).

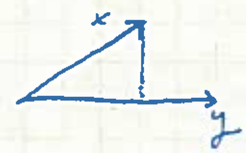
Defn The component of x along y is the scalar
$$c = \frac{\langle x, y\rangle}{\|y\|^2}.$$

The <u>orthogonal projection</u> of x along y is $cy = \dfrac{\langle x,y\rangle}{\|y\|^2}\, y$

<u>e.g.</u> $x=(3,2)$, $y=(5,0) \in \mathbb{R}^2$. Then

$$c = \frac{\langle x,y\rangle}{\|y\|^2} = \frac{(3,2)\cdot(5,0)}{(5,0)\cdot(5,0)} = \frac{15}{25} = \frac{3}{5}$$

and $cy = \frac{3}{5}(5,0) = (3,0)$



<u>Prop.</u> (1) $\|cx\| = |c|\,\|x\|$

(2) $\|x\| = 0$ iff $x=0$

(3) $|\langle x,y\rangle| \le \|x\|\,\|y\|$ (Cauchy-Schwartz)

(4) $\|x+y\| \le \|x\| + \|y\|$ (triangle)

<u>Pf</u> (1), (2) : exc.

(3): If $y=0$, done, so assume $y \neq 0$ and let $c = \dfrac{\langle x,y\rangle}{\|y\|^2}$.

Then $x-cy \perp y$ so, by Pythagoras,

$$\|x-cy\|^2 + \|cy\|^2 = \|x\|^2$$

$$\implies \|cy\|^2 \le \|x\|^2$$

$$\implies \|x\| \ge \|cy\| = |c|\,\|y\| = \frac{|\langle x,y\rangle|}{\|y\|}.$$

$$\implies \|x\|\|y\| \ge |\langle x,y\rangle|.$$

(4): $\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$

$= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2$

$= \|x\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) + \|y\|^2$ $\quad(z+\bar{z} = 2\,\mathrm{Re}(z))$

$\le \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$

$\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$ $\quad$ (CS)

$= (\|x\| + \|y\|)^2$ $\quad\square$

<u>Defn.</u> The <u>distance</u> between $x, y \in V$ is

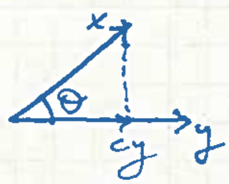
$$d(x,y) := \|x-y\| .$$

<u>Prop</u> (1) $d(x,y) = d(y,x)$

(2) $d(x,y) \geq 0$ with equality iff $x = y$

(3) $d(x,y) \leq d(x,z) + d(z,x)$. $\square$

<u>Angles</u> $(V, \langle , \rangle)$ inner product space over $F = \mathbb{R}$. (not $\mathbb{C}$)



<u>Defn.</u> The <u>angle</u> θ between $x, y \in V$ is

$$\theta = \arccos\left(\frac{\langle x,y \rangle}{\|x\| \, \|y\|} \right) .$$

Thus $\langle x,y \rangle = \|x\| \, \|y\| \cos\theta$.

<u>Rmk</u> • By C-S, $|\langle x,y \rangle| \leq \|x\| \, \|y\|$, so

$$-1 \leq \frac{\langle x,y \rangle}{\|x\| \, \|y\|} \leq 1$$ and arccos makes since.

• $\cos\theta = \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle$

unit vector in direction of x .