Recall the lin trans $\mathbb{R}^2 \to \mathbb{R}^2$ given by $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which rotates the plane by $\pi/2$. We have

$$p_A(x) = \det(A - xI) = \det\begin{pmatrix} -x & -1 \\ 1 & -x \end{pmatrix} = x^2+1$$

which has no roots in $\mathbb{R}$ and thus A has no eigenvalues.

Now consider the lin trans $\mathbb{C}^2 \to \mathbb{C}^2$ given by A.

Over $\mathbb{C}$, $p_A(x) = (x+i)(x-i)$ and A has eigenvalues $\pm i$.

__Prop__ If $\dim V = n$ and $f \in \mathcal{L}(V)$ has n __distinct__ eigenvalues, then f is diagonalizable.

__Pf__ TPS (know eigenvectors of distinct eigenvalues are lin ind) $\square$

Compute a basis for eigenspace of i:

$$A - iI_2 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \xrightarrow{r_2 \to r_2 + ir_1} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \ker(A - iI_2) = \{(iy, y) \mid y \in \mathbb{C}\} = \operatorname{span}\{(i,1)\}$.

Similarly, $\ker(A + iI_2) = \operatorname{span}\{(-i,1)\}$.

__Check__ $A\begin{pmatrix} i \\ 1 \end{pmatrix} = i\begin{pmatrix} i \\ 1 \end{pmatrix}$ $\quad A\begin{pmatrix} -i \\ 1 \end{pmatrix} = -i\begin{pmatrix} i \\ 1 \end{pmatrix}$.

So for $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, $P^{-1}AP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

__Defn__ A polynomial $p \in F[x]$ __splits over F__ if $\exists\, c, \lambda_1, \dots, \lambda_n \in F$ s.t. $p(x) = c(x-\lambda_1)\cdots(x-\lambda_n)$.

__Note__ The λ_i need not be distinct. The number of times a particular value λ occurs among the λ_i is called its __algebraic multiplicity__.

__Fundamental Thm of Algebra__ Every $p \in \mathbb{C}[x]$ splits over $\mathbb{C}$.

Recall $E_\lambda(f) = \ker(f - \lambda I)$ is the λ-eigenspace of f.

Call $\dim E_\lambda(f)$ the geometric multiplicity of λ.

Call multiplicity of λ as a root of $p_f(x)$ the algebraic multiplicity of λ.

We have already seen that f is diagonalizable iff geometric multiplicities add to $\dim V$.

Prop For $\dim V < \infty$, λ an eigenvalue of $f \in \mathcal{L}(V)$, the geometric multiplicity of λ is $\leq$ alg mult of λ.

Pf Take $v_1, \dots, v_k$ basis of $E_\lambda(f)$ and extend to $v_1, \dots, v_n$ basis of V. With respect to this basis, the matrix for f takes the form

$$A = \begin{pmatrix} \lambda I_k & B \\ 0 & C \end{pmatrix} \quad \text{where } B, C \in M_{(n-k)\times(n-k)}(F).$$

Thus

$$p_f(x) = \det \begin{pmatrix} (\lambda - x) I_k & B \\ 0 & C - xI_{n-k} \end{pmatrix}$$

$$= (\lambda - x)^k \det(C - xI_{n-k})$$

$$= (\lambda - x)^k g(x)$$

for some $g \in F[x]$. Thus $k \leq$ alg mult of λ. $\square$

Jordan Form

Suppose $p_f(x)$ splits over F, but f is not diagonalizable.
Does $\exists$ basis α s.t. $M_\alpha(f)$ is still "nice"?

Defn A Jordan block of size k for $\lambda \in F$ is the $k \times k$ matrix $J_k(\lambda)$ w/ λ's on diagonal and 1's on the "superdiagonal"

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

A matrix is in <u>Jordan form</u> if it is block diagonal with Jordan blocks for various λ along diagonal:

$$\begin{pmatrix} J_{k_1}(\lambda_1) & & & 0 \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{k_m}(\lambda_m) \end{pmatrix}$$

e.g.

$$\left(\begin{array}{cc|cc|c|ccc} 2 & 1 & & & & & & \\ & 2 & & & & & & \\ \hline & & 2 & 1 & & & & \\ & & & 2 & & & & \\ \hline & & & & 5 & & & \\ \hline & & & & & 4 & 1 & \\ & & & & & & 4 & 1 \\ & & & & & & & 4 \end{array}\right) = \begin{pmatrix} J_2(2) & & & \\ & J_2(2) & & \\ & & J_1(5) & \\ & & & J_3(4) \end{pmatrix}$$

$$=: J_2(2) \oplus J_2(2) \oplus J_1(5) \oplus J_3(4)$$

<u>Thm</u> Let $\dim V < \infty$. Suppose $f \in \mathcal{L}(V)$ and $p_f(x)$ splits over F. Then $\exists$ ord basis for V s.t. $M_\alpha(f)$ is in Jordan form. The Jordan form is unique up to permutation of the Jordan blocks.

<u>Pf</u> Math 332 via structure thm for fin gen modules over a principal ideal domain. $\square$

<u>TPS</u> · Determine $J_k(\lambda)^m$

· Use the Jordan form thm to show that every $f \in \mathcal{L}(V)$ with p_f split $/F$ has a matrix rep which is the sum $D+N$ of a diagonal matrix D and <u>nilpotent</u> matrix N (so that $N^r = 0$ for some $r \in \mathbb{N}$).