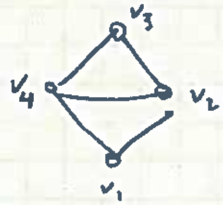
Walks on Graphs

For $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, $D^l = \mathrm{diag}(\lambda_1^l, \dots, \lambda_n^l)$.

So if $A = PDP^{-1}$, then (HW) $A^l = PD^lP^{-1}$ is easily computed.

A graph consists of vertices connected by edges:



4 vxs
5 edges

A walk of length l is a graph is a sequence of vxs $u_0, u_1, \dots, u_l$ in the graph with u_{i-1} connected to u_i by an edge for $i = 1, \dots, l$.

e.g. v_1v_4, $v_1v_2v_3v_4$ are walks from v_1 to v_4 of length 1 and length 3 above.

Defn Let G be a graph with vxs $v_1, \dots, v_n$. The adjacency matrix of G is the $n \times n$ matrix $A = A(G)$ defined by

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge connecting } v_i \text{ to } v_j \\ 0 & \text{o/w.} \end{cases}$$

e.g. $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ for the diamond graph.

Thm The number of walks of length l from v_i to v_j is $(A^l)_{ij}$.

Pf HW. $\square$

e.g. For $A = A(\text{diamond graph})$, $A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & \boxed{1} & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$, $A^3 = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & \boxed{4} & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}$

so, e.g., 1 path of length 2 from v_2 to v_3
4 paths of length 3 from v_2 to itself.

TPS Verify

Thm If $A \in M_{n\times n}(\mathbb{R})$ is symmetric ($A = A^T$), then A is diagonalizable over $\mathbb{R}$.

Pf Math 202 via spectral thm. $\square$

Note $A(G)$ is symmetric!

Thus $\exists$ diagonal D s.t. $P^{-1}AP = D$ for some P and $A^l = PD^lP^{-1}$.

It follows that the # of walks of length l b/w v_i, v_j is a linear combination of the l-th powers of the eigenvalues of A:

$$c_1\lambda_1^l + \cdots + c_n\lambda_n^l.$$

Defn A walk is closed if it begins and ends at the same vx.

Defn For $A \in M_{n\times n}(F)$, the trace of A is the sum of its diagonal entries, $tr(A) = \sum_{i=1}^{n} A_{ii}$.

Prop For $A = A(G)$, the number of closed walks in G of length l is $tr(A^l)$.

Pf The # of closed walks of length l from v_i to v_i is $(A^l)_{ii}$. Summing over $i = 1, \dots, n$ gives $tr(A^l)$. $\square$

Prop For $A \in M_{n\times n}(F)$ with $p_A(x) = c(x-\lambda_1)\cdots(x-\lambda_n)$,

$$tr(A) = \lambda_1 + \cdots + \lambda_n.$$

Note True even if the λ_i are in some larger field $K \supseteq F$ (and such a field always exists s.t. p_A splits).

Pf Take P s.t. $P^{-1}AP = J$ is in Jordan form. Then eigenvalues of A are on the diagonal of J, each appearing a # of times equal to its algebraic multiplicity. Fact $tr(UV) = tr(VU)$.

Thus $tr(A) = tr(PJP^{-1}) = tr(JPP^{-1}) = tr(J) = \lambda_1 + \cdots + \lambda_n$. $\square$

Cor For $A = A(G) \in M_{n\times n}(\mathbb{R})$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ the eigenvalues of A (with multiplicity), Then the # closed walks in G of length l is $\sum_{i=1}^{n} \lambda_i^l$.

Pf. $tr(A^\ell) = \sum$ eigenvalues of $A^\ell = \lambda_1^\ell + \cdots + \lambda_n^\ell$. $\square$

e.g. For the diamond graph,

$$\det(A - xI_4) = x^4 - 5x^2 - 4x = x(x+1)(x^2 - x - 4)$$

so eigenvalues are $0, -1, \frac{1 \pm \sqrt{17}}{2}$.

Thus the # closed walks in G of length ℓ is

$$w(\ell) = (-1)^\ell + \left(\frac{1+\sqrt{17}}{2}\right)^\ell + \left(\frac{1-\sqrt{17}}{2}\right)^\ell.$$

ℓ	1	2	3	4	5	6
$w(\ell)$	0	10	12	50	100	298