MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 9

Problem 1. Compute the determinant of

$$\begin{pmatrix} 2 & 2 & 1 \\ 3 & -1 & 0 \\ -2 & 0 & 5 \end{pmatrix}$$

via its permutation expansion, via cofactor expansion, and also via Gauss-Jordan reduction. Which took the fewest number of steps?

Problem 2 (Cramer's Rule). Suppose that $A \in M_{n\times n}(F)$ is invertible and that we want to solve the system of linear equations $Ax = b$, where $x = (x_1, \dots, x_n)$ and $b = (b_1, \dots, b_n)$ are regarded as column vectors. (Here, the x_i are variables and the b_i are scalars.)

(a) Assuming $Ax = b$, show that $(\det A)x = \mathrm{adj}(A)b$.
(b) Show that $x_i = \frac{\det(B_i)}{\det(A)}$, where B_i is obtained by replacing the i-th column of A with b.
(c) Use Cramer's Rule to solve the system

$$\begin{aligned} 2x - 3y &= 1, \\ x + 5y &= 4. \end{aligned}$$

Problem 3. Let

$$M = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}).$$

Check that $(1, 4, 2)$, $(1, 0, 1)$, and $(0, 1, 0)$ are linearly independent eigenvectors of M. Use this to determine a diagonal matrix to which M is similar ($M = PDP^{-1}$ for D diagonal).

Problem 4. Let $f : V \to V$ be a linear transformation. Set $f^0 = \mathrm{id}_V$ and for $m \geq 1$ set $f^m = f \circ f^{m-1}$ (so $f^1 = f$, $f^2 = f \circ f$, $f^3 = f \circ f \circ f$, *etc.*). Suppose that λ is an eigenvalue of f. Prove that λ^m is an eigenvalue of f^m for all $m \in \mathbb{N}$.

Problem 5. Let $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be the map $T(A) = A^T$, the transpose of A.

(a) Show that the only eigenvalues of T are ± 1. (*Hint*: Use the previous problem.)
(b) For $n = 2$, describe the eigenvectors corresponding to each eigenvalue.
(c) Find an ordered basis α for $M_{2\times 2}(\mathbb{R})$ such that the matrix representing T with respect to α is diagonal.
(d) [Bonus] Repeat parts (b) and (c) for an arbitrary $n > 2$.