MATH 201: LINEAR ALGEBRA
HOMEWORK DUE TUESDAY WEEK 8

Problem 1. Let V and W be finite-dimensional vector spaces and let $\varphi \in \mathcal{L}(V, W)$.
(a) Show that φ^* is injective if and only if φ is surjective.
(b) Show that φ^* is surjective if and only if φ is injective.

Problem 2. The *trace* of a square matrix is defined to be the sum of its diagonal entries. Let A be a 2×2 matrix with entries in a field. Prove that $\det(A + I) = \det(A) + \det(I)$ if and only if $\operatorname{trace}(A) = 0$.

Problem 3 (Vandermonde determinant)**.** Prove that the determinant of the matrix

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

is $(b - a)(c - a)(c - b)$.

Problem 4.
(a) A square matrix A is called *orthogonal* if $AA^t = I$, where A^t denotes the transpose of A. Show that if A is an orthogonal matrix, then $\det(A) = \pm 1$. Give an example of an orthogonal matrix A for which $\det(A) = -1$.
(b) A square matrix A over a field F is called *skew-symmetric* if $A^t = -A$. Show that if n is odd and $F = \mathbb{C}$, then no $n \times n$ skew-symmetric matrix A is invertible.