MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 8

Problem 1. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}.$$
Find elementary matrices $E_1, \ldots, E_\ell$ such that $E_\ell \cdots E_2 E_1 A$ is the reduced echelon form of A. (Check your work.)

In the next two exercises we will prove that the determinant is multiplicative, that is, that for $n \times n$ matrices A and B,
$$\det(AB) = \det(A)\det(B).$$

Problem 2. Let B be a fixed $n \times n$ matrix over F such that $\det(B) \neq 0$. Consider the function
$$d\colon M_{n\times n}(F) \longrightarrow F$$
defined by $d(A) = \det(AB)/\det(B)$. You will prove that $d(A) = \det(A)$. For a matrix A, we write $(r_1, \ldots, r_n)$ for the rows of A, with each $r_i \in F^n$.

(a) Prove that d is multilinear on rows, that is, d satisfies that
$$d(r_1, \ldots, r_i + k \cdot r_i', \ldots, r_n) = d(r_1, \ldots, r_i, \ldots, r_n) + kd(r_1, \ldots, r_i', \ldots, r_n)$$
for all $r_1, \ldots, r_n, r_i' \in F^n$ and any $k \in F$.

(Some suggested notation to help in your proof: let $c_1, \ldots, c_n$ be the columns of B. Then
$$(AB)_{s,t} = r_s \cdot c_t,$$
i.e., the s, t-entry of AB is the dot product of the s-th row of A with the t-th column of B. Recall that the dot product is defined by $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n$. Letting A' be the matrix with rows $(r_1, \ldots, r_i', \ldots, r_n)$ and A'' the matrix with rows $(r_1, \ldots, r_i + kr_i', \ldots, r_n)$, you will need compare the rows of AB, $A'B$ and $A''B$.)

(b) Prove that d is alternating on rows, that is, d satisfies that $d(r_1, \ldots, r_n) = 0$ if $r_i = r_j$ for some $i \neq j$.

(c) Prove that $d(I_n) = 1$.

(d) Deduce that for all A, we have that $d(A) = \det(A)$, and that $\det(AB) = \det(A)\det(B)$.

Problem 3. We still need to prove that $\det(AB) = \det(A)\det(B)$ when $\det(B) = 0$.

(a) Let $f\colon V \to W$ and $g\colon W \to U$ be linear transformations of finite dimensional vector spaces over F. Show that
$$\ker(f) \subseteq \ker(g \circ f) \qquad \text{and} \qquad \operatorname{im}(g \circ f) \subseteq \operatorname{im}(g).$$

(b) Use part (a) to prove that $\operatorname{rank}(g \circ f) \leq \operatorname{rank}(f)$ and $\operatorname{rank}(g \circ f) \leq \operatorname{rank}(g)$. (*Hint:* For one of them you might need to use the rank-nullity theorem.)

(c) Let A be an $m \times n$ matrix over F, and B an $n \times p$ matrix over F. Prove that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$ and $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

(d) Conclude that if both A and B are $n \times n$ matrices such that either $\det(A) = 0$ or $\det(B) = 0$, then $\det(AB) = 0$.