MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 7

Problem 1. Let $V = \mathbb{R}[x]_{\leq 5}$ be the $\mathbb{R}$-vector space of polynomials with coefficients in $\mathbb{R}$ in the variable x of degree at most 5 with $\{1, x, x^2, \ldots, x^5\}$ as basis. Prove that the following are elements of V^* and express them as linear combinations of the dual basis.

(a) $e : V \to \mathbb{R}$ defined by $e(p) = p(3)$ (*i.e.*, evaluation at $x = 3$).
(b) $\varphi : V \to \mathbb{R}$ defined by $\varphi(p) = \int_0^1 p(t)\, dt$.
(c) $\vartheta : V \to \mathbb{R}$ defined by $\vartheta(p) = p'(5)$, where p' denotes the usual derivative of p with respect to x.

Problem 2. Recall that if $\varphi : V \to W$ is a linear transformation, then $\varphi^* : W^* \to V^*$ is the linear transformation taking f to $f \circ \varphi$.

(a) Prove that if $\varphi : V \to W$ and $\psi : W \to U$ are linear transformations, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
(b) Use (a) and the theorem from class relating dual transformations and transpose matrices to prove that $(AB)^T = B^T A^T$ whenever $A \in M_{m\times p}(F)$ and $B \in M_{p\times n}(F)$.

Problem 3. For every subset S of a vector space V, show that $S^\circ = \{f \in V^* \mid f(s) = 0 \text{ for all } s \in S\}$ is a subspace of V^*.

Problem 4. Suppose that S is a subspace of V and let $i : S \to V$ denote the inclusion map given by $i(s) = s$. Prove that $\operatorname{im}(i^*) = S^*$.