MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 6

Problem 1. Let F be a field.

(a) Find $A, B \in M_{2\times 2}(F)$ that do not commute with each other under multiplication: $AB \neq BA$.
(b) A square matrix A is called *diagonal* if it is of the form aI for $a \in F$ and I the identity matrix. Show that if $A \in M_{2\times 2}(F)$ is diagonal, then $AB = BA$ for all $B \in M_{2\times 2}(F)$.
(c) [Bonus] The set of matrices $A \in M_{2\times 2}(F)$ commuting (under multiplication) with all $B \in M_{2\times 2}(F)$ is called the *center* of $M_{2\times 2}(F)$. In part (b), you showed that every diagonal matrix is in the center of $M_{2\times 2}(F)$. Prove the opposite inclusion, concluding that the center of $M_{2\times 2}(F)$ is the set of diagonal matrices.
(d) [Bonus] For a fixed $A \in M_{2\times 2}(F)$, the *centralizer* of A is the set of $B \in M_{2\times 2}(F)$ such that $AB = BA$ (*i.e.*, the centralizer of A is the set of matrices commuting with A). Determine the centralizer of A. *Hint*: The answer will depend on whether or not A is diagonal.

Problem 2. Let $\mathbb{R}[x]_{\leq n}$ be the vector space of polynomials in x with coefficients in $\mathbb{R}$ and degree at most n. Define $f : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 2}$ to be the linear transformation given by

$$f(p(x)) = (2 + x) \cdot p'(x) + 3p(x)$$

and define $g : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$ to be the linear transformation given by

$$g(a + bx + cx^2) = (a - b, c, a + b).$$

Let $B = \{1, x, x^2\}$ be an ordered basis for $\mathbb{R}[x]_{\leq 2}$ and let $E = \{e_1, e_2, e_3\}$ be the standard ordered basis for $\mathbb{R}^3$.

(a) Compute the matrix representing f with respect to the ordered basis B for both the domain and codomain.
(b) Compute the matrix representing g with respect to the ordered bases B and E.
(c) Compute the matrix representing $g \circ f$ with respect to the basis B and E. Then use Theorem 2.7 of Chapter Three, §IV to verify your result.

Problem 3. Using the notation of Problem 2, define

$$\begin{aligned} f : \mathbb{R}[x]_{\leq 2} &\to \mathbb{R}[x]_{\leq 3} \\ p &\mapsto \int_0^x p(t)\, dt. \end{aligned}$$

Verify that f is a linear transformation and compute the matrix representing f with respect to the ordered bases $\{1, x, x^2\}$ for $\mathbb{R}[x]_{\leq 2}$ and $\{1, x, x^2, x^3\}$ for $\mathbb{R}[x]_{\leq 3}$.

Problem 4. Use the algorithm from class to determine whether the following matrices are invertible; if invertible, write down the associated inverse matrix. (N.B. You can easily check if your answer is correct!)

$$A = \begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -5 & 3 \\ 9 & -5 & 2 \\ 4 & 0 & 1 \end{pmatrix}$$