MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 5

Problem 1. Let $V = F[x]_{\leq 2}$ be the vector space of polynomials with coefficients in F of degree ≤ 2. For $k \in F$, let $S_k : V \to V$ be the function taking $f(x)$ to $f(x-k)$. Check that S_k is a linear transformation and prove that it is an isomorphism.

Problem 2. Let V and W be F-vector spaces. Recall from Lemma 1.17 of Three.II.1 in the book that the set $\mathcal{L}(V, W)$ of linear transformations from V to W forms an F-vector space. The *dual* of V is defined to be the vector space $V^* := \mathcal{L}(V, F)$.

(a) Suppose that V is finite dimensional with basis $\{v_1, \ldots, v_n\}$. For $1 \leq i \leq n$, let $v_i^* \in V^*$ be the linear transformation such that $v_i^*(v_i) = 1$ and $v_i^*(v_j) = 0$ for $j \neq i$. Prove that $\{v_1^*, \ldots, v_n^*\}$ is a basis for V^* and conclude that $\dim V^* = n$ as well.

(b) [*Bonus*] By part (a), we see that $V \cong V^*$ when V is finite dimensional. Prove that $V \not\cong V^*$ for $V = \mathbb{R}[x]$.

Problem 3. Draw the image of the unit square $[0,1] \times [0,1] \subseteq \mathbb{R}^2$ under the following linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$, making sure to clearly label the image of e_1 and e_2 in each diagram. Also state what the rank and nullity of each map is (no proof required).

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (2x - y, x + y)$.

(b) $g : \mathbb{R}^2 \to \mathbb{R}^2$ the linear transformation taking $(1, 0)$ to $(1, 1)$ and $(0, 1)$ to $(-1, 1)$.

(c) $h = R_{\pi/4} \circ \pi_1$ where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\pi_1(x, y) = (x, 0)$ and $R_{\pi/4} : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation by $\pi/4$ map.

(d) $i = \pi_1 \circ R_{\pi/4}$.

(e) $j = \ell \circ k$ where k is reflection through the y-axis and ℓ is reflection through the line $y = -x$.