MATH 201: LINEAR ALGEBRA
HOMEWORK DUE TUESDAY WEEK 12

Throughout this assignment V denotes a vector space over $F = \mathbb{R}$ or $\mathbb{C}$.

Problem 1.

(a) Show that the function $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$f((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1 y_1 + x_3 y_3$$

is not an inner product on $\mathbb{R}^3$.

(b) Suppose that $\langle\ ,\ \rangle$ is an inner product on V. Prove that

$$\langle x, cy\rangle = \bar{c}\langle x, y\rangle \quad \text{for all } x, y \in V \text{ and } c \in F,$$

and

$$\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle \quad \text{for all } x, y, z \in V.$$

(c) Show that if $f : V \times V \to F$ is an inner product on V and $r \in \mathbb{R}_{>0}$, then the function $r \cdot f$ is also an inner product on V.

(d) Describe explicitly all inner products on $\mathbb{R}$ and on $\mathbb{C}$.

Problem 2.

(a) Suppose V is an inner product space. Prove the *parallelogram equality*

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 \quad \text{for all } u, v \in V.$$

(In the case $V = \mathbb{R}^2$ this equality expresses the fact that the sum of the squares of the sides of a parallelogram equals the sum of the squares of the diagonals.)

(b) A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

- $\|x\| = 0 \iff x = 0$;
- $\|cx\| = |c| \cdot \|x\|$ for all $c \in F$ and $x \in V$;
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Define a norm on $\mathbb{R}^n$ by $\|x\| = \max_i |x_i|$, where $x = (x_1, \ldots, x_n)$. Prove that this is indeed a norm, and that it does not come from an inner product; in other words, there does not exist an inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ such that $\|x\|^2 = \langle x, x\rangle$ for all $x \in \mathbb{R}^n$.

Problem 3. Use the Cauchy-Schwarz inequality to prove that

$$(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16$$

for all positive real numbers a, b, c, d.