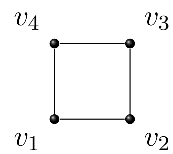
MATH 201 HOMEWORK ASSIGNMENT 20

Problem 1.

(a) Show that every 2×2 symmetric matrix over $\mathbb{R}$ is diagonalizable.

(b) Show that the complex symmetric matrix $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ is not diagonalizable.

Problem 2. Consider the *cycle graph* C_4:



(a) Find the adjacency matrix $A = A(G)$.

(b) Compute A^4 and use it to determine the number of walks from v_1 to v_3 of length 4. List all of these walks (these will be ordered lists of 5 vertices).

(c) What is the total number of *closed* walks of length 4?

(d) Compute and factor the characteristic polynomial of A.

(e) What are the algebraic multiplicities of each of the eigenvalues?

(f) Diagonalize A using our algorithm: compute bases for the eigenspaces of each of the eigenvalues you just found, and use them to construct a matrix P such that $P^{-1}AP$ is a diagonal matrix with the eigenvalues along the diagonal.

(g) Use part (f) to find a closed expression for A^ℓ for each $\ell \geq 1$.

(h) Take the trace of A^ℓ to get a formula for the number of closed walks of length ℓ for each $\ell \geq 1$.

Problem 3. In this problem you will prove the following theorem which was stated in class.

Theorem. *Let A be the adjacency matrix for a graph G with vertices $v_1, \ldots, v_n$, and let $\ell \in \mathbb{Z}_{\geq 0}$. Then the number of walks of length ℓ from v_i to v_j is $(A^\ell)_{ij}$.*

(a) Let $p(i, j, \ell)$ denote the number of walks of length ℓ in G from v_i to v_j. Prove that for all $i, j = 1, \ldots, n$ and $\ell \geq 1$,

$$p(i, j, \ell) = \sum_{k=1}^{n} p(i, k, \ell - 1) p(k, j, 1).$$

(*Hint:* Part of the trick is to parse this formula appropriately.)

(b) Prove the theorem by induction on ℓ, using the result from part (a).