MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 10

Note: Finish Problem 1 first.

Problem 1. Find the solution to the system of differential equations

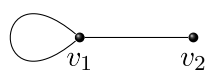
$$x_1' = x_1 + x_2$$
$$x_2' = x_1$$

with initial condition $(x_1(0), x_2(0)) = (1, 0)$. Your solution should include the diagonalization of a matrix. You may find the following notation useful:

$$\phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{\phi} = \frac{1-\sqrt{5}}{2},$$

with useful relations $\phi\overline{\phi} = -1$, $\phi^2 = \phi + 1$, and $\phi + \overline{\phi} = 1$. (**Warning:** You will want to make sure you get the diagonalization perfect. This will take some time. Using the above notation as much as possible will help.)

Problem 2. Find a closed form for the number of walks of length ℓ from v_1 to v_1 (closed walks at v_1) in the graph



Problem 3. Consider a sequence of numbers p_n defined recursively by fixing constants a and b, next assigning initial values for p_0 and p_1, and then for $n \geq 1$ letting

$$p_{n+1} = ap_n + bp_{n-1}.$$

For instance, letting $a = 2$, $b = -1$, $p_0 = 0$, and $p_1 = 1$, we get

$$p_0 = 0$$
$$p_1 = 1$$
$$p_{n+1} = 2p_n - p_{n-1} \quad \text{for } n \geq 1,$$

which defines the sequence

$$0, 1, 2, 3, 4, 5, 6, \ldots$$

Given any sequence of this form, we get the following matrix equation:

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}. \tag{1}$$

So we have

$$\begin{pmatrix} p_2 \\ p_1 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_0 \end{pmatrix},$$

which implies

$$\begin{pmatrix} p_3 \\ p_2 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ p_1 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \left[\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_0 \end{pmatrix} \right] = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} p_1 \\ p_0 \end{pmatrix},$$

and so on. In general, we have

(2) $$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} p_1 \\ p_0 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

and suppose A is diagonalizable. Take P so that

$$P^{-1}AP = D = \mathrm{diag}(\lambda_1, \lambda_2).$$

We have seen that it follows that $A^n = PD^nP^{-1}$, so that equation (2) becomes

$$\begin{pmatrix} p_{n+1} \\ p_n \end{pmatrix} = PD^nP^{-1} \begin{pmatrix} p_1 \\ p_0 \end{pmatrix} = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} \begin{pmatrix} p_1 \\ p_0 \end{pmatrix}.$$

Thus, we get a closed form expression for p_n in terms of powers of the eigenvalues of A (just take the second component of the product on the right-hand side of the above equation).

Let $a = b = 1$, $p_0 = 0$, and $p_1 = 1$.

(a) Write out the first several values for the sequence (p_n).
(b) Write the corresponding matrix equation, as above.
(c) Diagonalize the matrix A and compute the corresponding equation for p_n in terms of powers of the eigenvalues of A.