# MATH 201: LINEAR ALGEBRA
# HOMEWORK DUE TUESDAY WEEK 10

Let $F$ be a field and $V$ a finite-dimensional vector space over $F$.

**Problem 1.** Let $f \in \mathcal{L}(V)$. Suppose that $\alpha = \langle v_1, \ldots, v_n \rangle$ is a basis for $V$ such that $M_\alpha(f)$ is the diagonal matrix $\operatorname{diag}(c_1, \ldots, c_n)$. Prove the following.
(a) A basis for $\operatorname{im}(f)$ is $\{v_i : c_i \neq 0\}$, and $\operatorname{rank}(f) = \#\{i : c_i \neq 0\}$.
(b) A basis for $\ker(f)$ is $\{v_i : c_i = 0\}$, and $\operatorname{null}(f) = \#\{i : c_i = 0\}$.
(c) The determinant of $f$ is $c_1 \cdots c_n$.

**Problem 2.** Let $U_1, \ldots, U_k$ be subspaces of $V$. Suppose that $V = U_1 \oplus \cdots \oplus U_k$ and let $B_i$ be a basis for $U_i$. Prove the following.
(a) $B_i \cap B_j = \emptyset$ if $i \neq j$.
(b) The set $B = B_1 \cup \cdots \cup B_k$ is a basis for $V$.
(c) $\dim V = \dim U_1 + \cdots + \dim U_k$.

**Problem 3.** Let $A \in M_{n\times n}(F)$. Suppose that $A$ is diagonalizable, and let $\langle v_1, \ldots, v_n \rangle$ be a basis of $F^n$ consisting of eigenvectors of $A$. Let $P$ be the matrix with columns $v_1, \ldots, v_n$. Prove that the matrix $P^{-1}AP$ is diagonal.

**Problem 4.** Let $A \in M_{3\times 3}(\mathbb{R})$ be the matrix shown below.

$$A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$

(a) Compute the characteristic polynomial of $A$ and determine the eigenvalues of $A$.
(b) For each eigenvalue of $A$, determine the dimension of the associated eigenspace.
(c) Show that $A$ is diagonalizable and find a matrix $P$ such that $P^{-1}AP$ is diagonal.