MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 10

Problem 1. Let V denote the $\mathbb{R}$-vector space $\mathbb{R}[x]_{\leq 3}$ of polynomials in x with real coefficients and degree at most 3. Define a linear operator

$$\begin{aligned} L : V &\longrightarrow V \\ f &\longmapsto xf' + f' \end{aligned}$$

where f' denotes the usual derivative of f.

(a) Write the matrix of L with respect to the basis $\{1, x, x^2, x^3\}$ of V.
(b) What are the eigenvalues of L?
(c) Does V have a basis of eigenvectors of L? If so, give such a basis (written as polynomials, not tuples of real numbers), and if not, explain why not.

Problem 2. Suppose that V is an F-vector space and $f \in \mathcal{L}(V)$. Suppose further that $p_f(x)$ splits completely over F with no multiple roots (*i.e.*, its roots are distinct). Prove that f is diagonalizable.

Problem 3. For which a, b is the matrix

$$\begin{pmatrix} -1 & a & b \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

diagonalizable?

Problem 4. Let $A = \begin{pmatrix} a & 0 \\ 3(a-b) & b \end{pmatrix}$.

(a) Find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.
(b) Use your answer to (a) to produce a concise formula for A^k, k a positive integer.