MATH 113: DISCRETE STRUCTURES
MONDAY WEEK 8 HANDOUT

Recall that C_n is the number of unlabeled full binary trees with $n+1$ leaves, and that

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

for $n \geq 0$ (and $C_0 = 1$).

Problem 1. A *Dyck path* of length $2n$ is a monotonic lattice path in $[0,n]^2$ starting from $(0,0)$ and ending at (n,n) which never goes above the diagonal. Prove that there are C_n Dyck paths of length $2n$. (*Hint*: Let D_n be the number of such Dyck paths. Show that D_n satisfies the same recurrence as C_n.)

Dyck paths also give a proof of the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Recall that there are $\binom{2n}{n}$ monotonic lattice paths from $(0,0)$ to (n,n). We aim to partition the monotonic paths into $n+1$ subsets of equal size, where precisely one of the subsets is the collection of Dyck paths. This will prove that $C_n = \binom{2n}{n}/(n+1)$, as desired.

We define the *exceedance* of a monotonic lattice path to be its number of vertical steps above the diagonal. The exceedance of a monotonic lattice path from $(0,0)$ to (n,n) is between 0 and n (inclusive), and the Dyck paths are precisely those monotonic lattice paths with exceedance 0. Let P be the set of monotonic lattice paths from $(0,0)$ to (n,n) and let E_i be the set of paths with exceednace i; then $P = E_0 \cup E_1 \cup \cdots \cup E_n$ is clearly a partition of P. If we can show that $|E_0| = |E_1| = |E_2| = \cdots = |E_n|$, then we will be done.

Given a path $p \in E_i$, write $p = BrAuC$ where r is the first right step below the diagonal and u is the first up step touching the diagonal after r. Then B is a path above the diagonal with exceedance $j \leq i$, A is a path below the diagonal, and C is the reamining path with exceedance $i-j$. Switch Br and Au to produce $f(p) = AuBrC$. The exceedances of A, uBr, and C are 0, $j+1$, and $i-j$, respectively. (Draw some pictures and check this!) Thus $f(p) \in E_{i+1}$.

Given a path $q \in E_{i+1}$, write $q = AuBrC$ where u is the first up step above the diagonal and r is the first right step touching the diagonal after u. Define $g(q) = BuAdC$ and check that $g(q) \in E_i$. Finally, check that $f : E_i \to E_{i+1}$ and $g : E_{i+1} \to E_i$ are inverse to each other. □

Problem 2. Prove that we can also express C_n as

$$C_n = \frac{(2n)!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n+1}$$