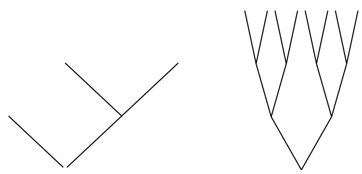
MATH 113: DISCRETE STRUCTURES
MONDAY WEEK 8 HANDOUT

A *full binary tree* is a rooted tree in which each vertex has either two children or no children; furthermore, when there are two children, one is designated *left* and the other *right*. Vertices with no children are called *leaves*.

Here are some examples:



Problem 1. Let C_n denote the number of unlabelled full binary trees with $n+1$ leaves. Prove that $C_0 = 1$ and

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

for $n \geq 0$. Compute the first several values of C_n and draw the corresponding full binary trees.

The numbers C_n are called the *Catalan numbers* and can be expressed concisely as $C_n = \frac{1}{n+1}\binom{2n}{n}$. The shortest proof I know of this fact uses *generating functions* and will not be presented here. A bijective proof for this formula appears after we establish that some other combinatorial structures are counted by Catalan numbers as well.

Problem 2. Find an explicit bijection between full binary trees with $n+1$ leaves and full parenthesizations of $n+1$ factors. (For instance, the full parenthesizations of abc are $(ab)c$ and $a(bc)$, while the full parenthesizations of $abcd$ are $((ab)c)d$, $(a(bc))d$, $(ab)(cd)$, $a((bc)d)$, and $a(b(cd))$.) This proves that C_n counts the number of full parenthesizations of $n+1$ factors.

It follows that C_n is also the number of ways of arranging n pairs of correctly matched parentheses. This perspective is very important in computer science, where trees are frequently stored via bracketing schemes.

Problem 3. A *Dyck path* of length $2n$ is a monotonic lattice path in $[0,n]^2$ starting from $(0,0)$ and ending at (n,n) which never goes above the diagonal. Prove that there are C_n Dyck paths of length $2n$. (*Hint*: Let D_n be the number of such Dyck paths. Show that D_n satisfies the same recurrence as C_n.)

Dyck paths also give a proof of the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n+1}.$$

Proof. Recall that there are $\binom{2n}{n}$ monotonic lattice paths from $(0,0)$ to (n,n). We aim to partition the monotonic paths into $n+1$ subsets of equal size, where precisely one of the subsets is the collection of Dyck paths. This will prove that $C_n = \binom{2n}{n}/(n+1)$, as desired.

We define the *exceedance* of a monotonic lattice path to be its number of vertical steps above the diagonal. The exceedance of a monotonic lattice path from $(0,0)$ to (n,n) is between 0 and n (inclusive), and the Dyck paths are precisely those monotonic lattice paths with exceedance

0. Let P be the set of monotonic lattice paths from $(0,0)$ to (n,n) and let E_i be the set of paths with exceednace i; then $P = E_0 \cup E_1 \cup \cdots \cup E_n$ is clearly a partition of P. If we can show that $|E_0| = |E_1| = |E_2| = \cdots = |E_n|$, then we will be done.

Given a path $p \in E_i$, write $p = BrAuC$ where r is the first right step below the diagonal and u is the first up step touching the diagonal after r. Then B is a path above the diagonal with exceedance $j \leq i$, A is a path below the diagonal, and C is the reamining path with exceedance $i - j$. Switch Br and Au to produce $f(p) = AuBrC$. The exceedances of A, uBr, and C are 0, $j+1$, and $i-j$, respectively. (Draw some pictures and check this!) Thus $f(p) \in E_{i+1}$.

Given a path $q \in E_{i+1}$, write $q = AuBrC$ where u is the first up step above the diagonal and r is the first right step touching the diagonal after u. Define $g(q) = BuAdC$ and check that $g(q) \in E_i$. Finally, check that $f : E_i \to E_{i+1}$ and $g : E_{i+1} \to E_i$ are inverse to each other. $\square$