MATH 113: DISCRETE STRUCTURES
WEDNESDAY WEEK 3 HANDOUT

Recall that for natural numbers n, k, the number

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!},$$

read "n choose k," is the number size k subsets of an n-element set. If $n \geq k$, this can also be written as $\frac{n!}{k!(n-k)!}$.

Problem 1. Compute the sums

$$\binom{1}{0}$$
$$\binom{2}{0} + \binom{2}{2}$$
$$\binom{3}{0} + \binom{3}{2}$$
$$\binom{4}{0} + \binom{4}{2} + \binom{4}{4}$$
$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4}$$
$$\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6}$$
$$\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6}$$

and develop a conjecture regarding the value of

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

where the sum's final term is $\binom{n}{n-1}$ or $\binom{n}{n}$ depending on whether n is odd or even, respectively. Give a combinatorial argument proving that your conjecture is true.

Problem 2. Compute the sums

$$\binom{0}{0}^2$$

$$\binom{1}{0}^2 + \binom{1}{1}^2$$

$$\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2$$

$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2$$

$$\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$$

$$\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2$$

by hand and develop a conjecture regarding the value of

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2.$$

Give a combinatorial argument proving that your conjecture is true.

Problem 3. How many ways are there to write a nonnegative integer m as a sum of r positive integer summands? (We decree that the order of the addends matters, so $3+1$ and $1+3$ are two different representations of 4 as a sum of 2 nonnegative integers.) Develop a conjecture and prove it.